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Stacks of Ramified Covers Under Diagonalizable Group Schemes

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Given a flat, finite group scheme G finitely presented over a base scheme we introduce the notion of ramified Galois cover of group G (or simply G-cover), which generalizes the notion of G-torsor. We study the stack of G-covers, denoted with G-Cov, mainly in the abelian case, precisely when G is a finite diagonalizable group scheme over $\mathbb{Z}$. In this case, we prove that G-Cov is connected, but it is irreducible or smooth only in few finitely many cases. On the other hand, it contains a "special" irreducible component $\mathcal{Z}_G$, which is the closure of $\mathrm{B}G$ and this reflects the deep connection we establish between G-Cov and the equivariant Hilbert schemes. We introduce "parametrization" maps from smooth stacks, whose objects are collections of invertible sheaves with additional data, to $\mathcal{Z}_G$ and we establish sufficient conditions for a G-cover in order to be obtained (uniquely) through those constructions. Moreover, a toric description of the smooth locus of $\mathcal{Z}_G$ is provided.

1 Introduction

Let G be a flat, finite group scheme finitely presented over a base scheme (say over a field, or, as in this paper, over $\mathbb{Z}$). In this paper, we study G-Galois covers of very general schemes. We define a (ramified) G-*cover* as a finite morphism $f\colon X \longrightarrow Y$ with an action

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of G on X such that f is G-invariant and $f_*\mathcal{O}_X$ is fppf-locally isomorphic to the regular representation $\mathcal{O}_Y[G]$ as $\mathcal{O}_Y[G]$-comodule. This definition is somehow the most natural: it generalizes the notion of G-torsors and, under suitable hypothesis, coincides with the usual definition of Galois cover when the group G is constant (see, e.g., [2, 6, 18]). Moreover, as explained below, in the abelian case G-covers are tightly related to the theory of equivariant Hilbert schemes (see, e.g., [1, 10, 16, 19]).

We call G-Cov the stack of G-covers and the aim of this article will be to describe its structure, especially in the abelian diagonalizable case. Our first result is the following theorem:

Theorem (2.2, 2.10)**.** The stack G-Cov is algebraic and finitely presented over S. Moreover, BG, the stack of G-torsors, is an open substack of G-Cov. □

We denote by μ_n the diagonalizable group over $\mathbb{Z}$ associated to $\mathbb{Z}/n\mathbb{Z}$. In many concrete problems, one is interested in a more direct and concrete description of a G-cover $f: X \longrightarrow Y$. This is very simple and well known when $G = \mu_2$: such a cover f is given by an invertible sheaf $\mathcal{L}$ on Y with a section of $\mathcal{L}^{\otimes 2}$. Similarly, when $G = \mu_3$, a μ_3-cover f is given by a pair $(\mathcal{L}_1, \mathcal{L}_2)$ of invertible sheaves on Y with maps $\mathcal{L}_1^{\otimes 2} \longrightarrow \mathcal{L}_2$ and $\mathcal{L}_2^{\otimes 2} \longrightarrow \mathcal{L}_1$ (see [3, §6]).

In general, however, there is no comparable description of G-covers. Very little is known when G is not abelian, beyond the cases $G = S_d$ with $d = 3, 4, 5$: see [6] for the case $G = S_3$ and [4, 5, 9, 15] for the non-Galois case.

Even in the abelian case, the situation becomes complicated very quickly when the order of G grows. The paper that inspires our work is [18]; here, the author describes G-covers $X \longrightarrow Y$ when G is an abelian group, Y is a smooth variety over an algebraically closed field of characteristic prime to $|G|$ and X is normal, in terms of certain invertible sheaves on Y, generalizing the description given above for $G = \mu_2$ and μ_3.

Here, we concentrate on the case when G is a finite diagonalizable group scheme over $\mathbb{Z}$; thus, G is isomorphic to a finite direct product of group schemes of the form μ_d for $d \geq 1$. We consider the dual finite abelian group $M = \mathrm{Hom}(G, \mathbb{G}_m)$ so that, by standard duality results (see [8]), G is the fppf sheaf of homomorphisms $M \longrightarrow \mathbb{G}_m$ and a decomposition of M into a product of cyclic groups yields the decomposition of G into a product of μ_d's.

In this case, we have an explicit description of a G-cover in terms of sequences of invertible sheaves. Indeed a G-cover over Y is of the form $X = \mathrm{Spec}\,\mathcal{A}$, where $\mathcal{A}$ is a coherent sheaf of algebras over Y with a decomposition

$$\mathscr{A} = \bigoplus_{m\in M} \mathcal{L}_m \text{ s.t. } \mathcal{L}_0 = \mathcal{O}_Y, \mathcal{L}_m \text{ invertible and } \mathcal{L}_m\mathcal{L}_n \subseteq \mathcal{L}_{m+n} \text{ for any } m, n \in M.$$

So a G-cover corresponds to a sequence of invertible sheaves $(\mathcal{L}_m)_{m\in M}$ with maps $\psi_{m,n}$: $\mathcal{L}_m \otimes \mathcal{L}_n \longrightarrow \mathcal{L}_{m+n}$ satisfying certain rules and our principal aim will be to simplify the data necessary to describe such covers. For instance, G-torsors correspond to sequences where all the maps $\psi_{m,n}$ are isomorphisms. Therefore, if $G = \mu_l$, a G-torsor is simply given by an invertible sheaf $\mathcal{L} = \mathcal{L}_1$ and an isomorphism $\mathcal{L}^{\otimes l} \simeq \mathcal{O}$.

When $G = \mu_2$ or $G = \mu_3$ the description given above shows that the stack G-Cov is smooth, irreducible, and very easy to describe. In the general case, its structure turns out to be extremely intricate. For instance, as we will see, G-Cov is almost never irreducible, but has a “special” irreducible component, called $\mathcal{Z}_G$, which is the scheme-theoretically closure of BG. This parallels what happens in the theory of M-equivariant Hilbert schemes (see [10, Remark 5.1]). It turns out that this theory and the theory of G-covers are deeply connected: given an action of G on $\mathbb{A}^r$, induced by elements $\underline{m} = m_1, \dots, m_r \in M$, the equivariant Hilbert scheme M-Hilb $\mathbb{A}^r$, which we will denote by M-Hilb$^{\underline{m}}$ to underline the dependency on the sequence $\underline{m}$, can be viewed as the functor whose objects are G-covers with an equivariant closed immersion in $\mathbb{A}^r$. The forgetful map $\vartheta : M\text{-Hilb}^{\underline{m}} \longrightarrow G\text{-Cov}$ is smooth and an atlas provided that $\underline{m}$ contains all the elements in $M - \{0\}$ (4.8). Moreover, $\vartheta^{-1}(\mathcal{Z}_G)$ coincides with the main component of M-Hilb$^{\underline{m}}$, first studied by Nakamura in [16].

We will prove the following results on the structure of G-Cov.

Theorem (4.13, 4. 17, 4.18, 4.20). When G is a finite diagonalizable group scheme over $\mathbb{Z}$, the stack G-Cov is

- flat and of finite type with geometrically connected fibers,
- smooth if and only if $G \simeq \mu_2, \mu_3, \mu_2 \times \mu_2$,
- normal if $G \simeq \mu_4$,
- reducible if $|G| \geq 8$ and $G \not\simeq (\mu_2)^3$.

The above properties continue to hold if we replace G-Cov with M-Hilb$^{\underline{m}}$ if $M - \{0\} \subseteq \underline{m}$. □

We do not know whether G-Cov is integral for $G \simeq \mu_5, \mu_6, \mu_7, (\mu_2)^3$. So G-Cov is usually reducible, its structure is extremely complicated and we have little hope of getting to a real understanding of the components not containing BG. Therefore, we will focus on the main irreducible component $\mathcal{Z}_G$ of G-Cov. The main idea behind this paper, inspired by the results in [18], is to try to decompose the multiplications $\psi_{m,n} \in \mathcal{L}_{m+n} \otimes \mathcal{L}_m^{-1} \otimes \mathcal{L}_n^{-1}$ as a tensor product of sections of other invertible sheaves. Following this idea, we will construct parametrization maps $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_G \subseteq G$-Cov, where $\mathcal{F}_{\underline{\mathcal{E}}}$ are "nice" stacks, for example smooth and irreducible, whose objects are those decompositions.

This construction can be better understood locally, where a G-cover over $Y = \operatorname{Spec} R$ is just $X = \operatorname{Spec} A$, where A is an R-algebra with an R-basis $\{v_m\}_{m\in M}$, $v_0 = 1$ ($\mathcal{L}_m = \mathcal{O}_Y v_m$), so that the multiplications are elements $\psi_{m,n} \in R$ such that $v_m v_n = \psi_{m,n} v_{m+n}$.

Consider $a \in R$, a collection of natural numbers $\mathcal{E} = (\mathcal{E}_{m,n})_{m,n\in N}$ and set $\psi_{m,n} = a^{\mathcal{E}_{m,n}}$. The condition that the product structure on $A = \bigoplus_m Rv_m$ defined by the $\psi_{m,n}$ yields an associative, commutative R-algebra, that is, makes Spec A into a G-cover over Spec R, translates in some additive relations on the numbers $\mathcal{E}_{m,n}$. Call $K_+^\vee$ the set of such collections $\mathcal{E}$. More generally, given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$, we can define a parametrization

$$R^r \ni (a_1, \dots, a_r) \longrightarrow \psi_{m,n} = a_1^{\mathcal{E}^1_{m,n}} \cdots a_r^{\mathcal{E}^r_{m,n}}.$$

This is essentially the local behavior of the map $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow G$-Cov. In the global case, the elements a_i will be sections of invertible sheaves.

From this point of view the natural questions are: given a G-cover over a scheme Y when does there exist a lift to an object of $\mathcal{F}_{\underline{\mathcal{E}}}(Y)$? Is this lift unique? How can we choose the sequence $\underline{\mathcal{E}}$?

The key point is to give an interpretation to $K_+^\vee$ (that also explains this notation). Consider $\mathbb{Z}^M$ with canonical basis $(e_m)_{m\in M}$ and define $v_{m,n} = e_m + e_n - e_{m+n} \in \mathbb{Z}^M/\langle e_0\rangle$. If $p : \mathbb{Z}^M/\langle e_0\rangle \longrightarrow M$ is the map $p(e_m) = m$, the $v_{m,n}$ generate $\operatorname{Ker} p$. Now call K_+ the submonoid of $\mathbb{Z}^M/\langle e_0\rangle$ generated by the $v_{m,n}$, $K = \operatorname{Ker} p$ its associated group and also consider the torus $\mathcal{T} = \underline{\operatorname{Hom}}(\mathbb{Z}^M/\langle e_0\rangle, \mathbb{G}_m)$, which acts on $\operatorname{Spec} \mathbb{Z}[K_+]$. By construction, we have that a collection of natural numbers $(\mathcal{E}_{m,n})_{m,n\in M}$ belongs to $K_+^\vee$ if and only if the association $v_{m,n} \longrightarrow \mathcal{E}_{m,n}$ defines an additive map $K_+ \longrightarrow \mathbb{N}$. Therefore, as the symbol suggests, we can identify $K_+^\vee$ with $\operatorname{Hom}(K_+, \mathbb{N})$, the dual monoid of K_+. Its elements will be called *rays*. More generally, a monoid map $\psi : K_+ \longrightarrow (R, \cdot)$, where R is a ring, yields a multiplication $\psi_{m,n} = \psi(v_{m,n})$ on $\bigoplus_{m\in M} Rv_m$ and therefore we obtain a map $\operatorname{Spec} \mathbb{Z}[K_+] \longrightarrow \mathcal{Z}_G$. We will prove that (see 4.6):

Theorem. We have $\mathcal{Z}_G \simeq [\operatorname{Spec} \mathbb{Z}[K_+]/\mathcal{T}]$ and B$G \simeq [\operatorname{Spec} \mathbb{Z}[K]/\mathcal{T}]$. □

We introduce the following notation: given $\alpha \in \mathbb{N}$, we set $0^{\alpha} = 1$ if $\alpha = 0$ and $0^{\alpha} = 0$ otherwise. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^{\vee}$ we have defined a map $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_G$. Note that if $\underline{\gamma}$ is a subsequence of $\underline{\mathcal{E}}$ then $\mathcal{F}_{\underline{\gamma}}$ is an open substack of $\mathcal{F}_{\underline{\mathcal{E}}}$ and $(\pi_{\underline{\mathcal{E}}})_{|\mathcal{F}_{\underline{\gamma}}} = \pi_{\underline{\gamma}}$. The lifting problem for the maps $\pi_{\underline{\mathcal{E}}}$ clearly depends on the choice of the sequence $\underline{\mathcal{E}}$. Considering larger $\underline{\mathcal{E}}$ allows us to parametrize more covers, but also makes uniqueness of the lifting unlikely. In this direction, we have proved that:

Theorem (3.21)**.** Let k be an algebraically closed field and suppose we have a collection $\underline{\mathcal{E}}$ whose rays generate the rational cone $K_+^{\vee} \otimes \mathbb{Q}$. Then $\mathcal{F}_{\underline{\mathcal{E}}}(k) \longrightarrow \mathcal{Z}_G(k)$ is essentially surjective. In other words, a G-cover of $\operatorname{Spec} k$ in the main component $\mathcal{Z}_G$ has a multiplication of the form $\psi_{m,n} = 0^{\mathcal{E}_{m,n}}$ for some $\mathcal{E} \in K_+^{\vee}$. $\square$

On the other hand, small sequences $\underline{\mathcal{E}}$ can guarantee uniqueness but not existence. The solution we have found is to consider a particular class of rays, called extremal, that have minimal nonempty support. Set $\underline{\eta}$ for the sequence of all extremal rays (that is finite). Note that extremal rays generate $K_+^{\vee} \otimes \mathbb{Q}$. We prove that:

Theorem (3.46, 3.47)**.** The smooth locus $\mathcal{Z}_G^{\text{sm}}$ of $\mathcal{Z}_G$ is of the form $[X_G/\mathcal{T}]$, where X_G is a smooth toric variety over $\mathbb{Z}$ (whose maximal torus is $\operatorname{Spec} \mathbb{Z}[K]$). Moreover, $\pi_{\underline{\eta}} : \mathcal{F}_{\underline{\eta}} \longrightarrow \mathcal{Z}_G$ induces an isomorphism of stacks

$$\pi_{\underline{\eta}}^{-1}(\mathcal{Z}_G^{\text{sm}}) \xrightarrow{\simeq} \mathcal{Z}_G^{\text{sm}}. \qquad \square$$

Among the extremal rays there are special rays, called smooth, that can be defined as extremal rays $\mathcal{E}$ whose associated multiplication $\psi_{m,n} = 0^{\mathcal{E}_{m,n}}$ yields a cover in $\mathcal{Z}_G^{\text{sm}}$. Set $\underline{\xi}$ for the sequence of smooth extremal rays. It turns out that the theorem above holds if we replace $\underline{\eta}$ with $\underline{\xi}$.

If, given a scheme X, we denote by $\underline{\operatorname{Pic}}\, X$ the category whose objects are invertible sheaves on X and whose arrows are arbitrary maps of sheaves, we also have:

Theorem (3.51)**.** Consider a 2-commutative diagram:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] & \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\
X \arrow[r] \arrow[ur, dashed] & G\text{-Cov}
\end{tikzcd}$$

where X and Y are schemes and $\underline{\mathcal{E}}$ is a sequence of elements of $K_+^\vee$. If $\underline{\text{Pic}}\, X \xrightarrow{f^*} \underline{\text{Pic}}\, Y$ is fully faithful (resp. an equivalence) the dashed lifting is unique (resp. exists and is unique). □

In particular, the theorems above allow us to conclude that:

Theorem (3.47, 3.52)**.** Let X be a locally noetherian and a locally factorial scheme. A cover $\chi \in G\text{-Cov}(X)$ such that $\chi_{|k(p)} \in \mathcal{Z}_G^{\text{sm}}(k(p))$ for any $p \in X$ with $\text{codim}_p X \leq 1$ lifts uniquely to $\mathcal{F}_{\underline{\xi}}(X)$. □

An interesting problem is to describe all (smooth) extremal rays. This seems very difficult and it is related to the problem of finding $\mathbb{Q}$-linearly independent sequences among the $v_{m,n} \in K_+$. A natural way of obtaining extremal rays is trying to describe G-covers with special properties. The first examples of them arise looking at covers with normal total space. Indeed in [18] the author is able to describe the multiplications yielding regular G-covers of a discrete valuation ring. This description, using the language introduced above, yields a sequence $\underline{\delta} = (\mathcal{E}^\phi)_{\phi \in \Phi_M}$ of smooth extremal rays, where Φ_M is the set of surjective maps $M \longrightarrow \mathbb{Z}/d\mathbb{Z}$ with $d > 1$. In this paper, we will define a stratification of G-Cov by open substacks $\text{B}G = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|G|-1} = G\text{-Cov}$ and we will prove that there exists an explicitly given sequence $\underline{\mathcal{E}}$ of smooth extremal rays (defined in Proposition 5.40) containing $\underline{\delta}$ such that:

Theorem (Theorems 4.40, 5.42)**.** We have $U_2 \subseteq \mathcal{Z}_G^{\text{sm}}$ and $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_G$ induces isomorphisms of stacks

$$\pi_{\underline{\mathcal{E}}}^{-1}(U_2) \xrightarrow{\simeq} U_2,\ \pi_{\underline{\delta}}^{-1}(U_1) = \pi_{\underline{\mathcal{E}}}^{-1}(U_1) \xrightarrow{\simeq} U_1.$$ □

The above theorem implies that M-Hilb$\mathbb{A}^2$ is smooth and irreducible (5.43). In this way, we get an alternative proof of the result in [13] (later generalized in [14]) in the particular case of equivariant Hilbert schemes.

Theorem (4.41, 5.45)**.** Let X be a locally noetherian and a locally factorial scheme and $\chi \in G\text{-Cov}(X)$. If $\chi_{|k(p)} \in U_1$ for any $p \in X$ with $\text{codim}_p X \leq 1$, then χ lifts uniquely to $\mathcal{F}_{\underline{\delta}}(X)$. If $\chi_{|k(p)} \in U_2$ for any $p \in X$ with $\text{codim}_p X \leq 1$, then χ lifts uniquely to $\mathcal{F}_{\underline{\mathcal{E}}}(X)$. □

Note that $\underline{\mathcal{E}} = \underline{\delta}$ if and only if $G \simeq (\mu_2)^l$ or $G \simeq (\mu_3)^l$ (Proposition 5.44). Finally we prove:

Theorem (Theorems 4.42, 5.55)**.** Let X be a locally noetherian and locally factorial integral scheme with $\dim X \geq 1$ and $(\operatorname{char} X, |M|) = 1$ and Y/X be a G-cover. If Y is regular in codimension 1 it is normal and Y/X comes from a unique object of $\mathcal{F}_{\underline{\delta}}(X)$. If Y is normal crossing in codimension 1 (see Definition 5.47) then Y/X comes from a unique object of $\mathcal{F}_{\underline{\gamma}}(X)$, where $\underline{\delta} \subseteq \underline{\gamma} \subseteq \underline{\mathcal{E}}$ is an explicitly given sequence. $\square$

The part concerning regular in codimension 1 covers is essentially a rewriting of [18, Theorem 2.1 and Corollary 3.1] extended to locally noetherian and locally factorial schemes, while the last part generalizes [2, Theorem 1.9].

Outline of the paper. We now briefly summarize how this paper is divided. In Section 2, we will introduce the notion of G-covers, for a general group G, and prove some facts about them, for example, the algebraicity of G-Cov. In Section 3 we will introduce some general tools that will be applied in the study of G-Cov, when G is a finite and diagonalizable group scheme. In this case, G-Cov and some of its substacks, like $\mathcal{Z}_G$ and BG, share a common structure, that is, they are all of the form $\mathcal{X}_\phi = [\operatorname{Spec} \mathbb{Z}[T_+]/\mathcal{T}]$, where T_+ is a finitely generated commutative monoid whose associated group is free of finite rank, $\mathcal{T}$ is a torus over $\mathbb{Z}$ and $\phi : T_+ \longrightarrow \mathbb{Z}^r$ is an additive map, that induces the action of $\mathcal{T}$ on $\operatorname{Spec} \mathbb{Z}[T_+]$. Section 3 will be dedicated to the study of such stacks. As we will see many facts about G-Cov are just applications of general results about such stacks. For instance the existence of a special irreducible component $\mathcal{Z}_\phi$ of $\mathcal{X}_\phi$ as well as the use of $T_+^\vee = \operatorname{Hom}(T_+, \mathbb{N})$ for the study of the smooth locus of $\mathcal{Z}_\phi$ are properties that can be stated in this setting. Section 4 and 5 are dedicated to the study of G-covers, in the particular case where G is a finite and diagonalizable group scheme. In Section 4, we will explain how G-Cov can be viewed as a stack of the form $\mathcal{X}_\phi$ and how it is related to the equivariant Hilbert schemes. Then we will study the properties of connectedness, irreducibility and smoothness for G-Cov. Finally, we will introduce the stratification $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|G|-1} = G$-Cov and we will characterize the locus U_1. In Section 5, we will study the locus U_2 and G-covers whose total space is normal crossing in codimension 1.

All the other sections will be dedicated to the study of G-Cov when G is a finite diagonalizable group with dual group $M = \operatorname{Hom}(G, \mathbb{G}_m)$.

Notation. A map of schemes $f : X \longrightarrow Y$ will be called a *cover* if it is finite, flat and of finite presentation or, equivalently, if it is affine and $f_*\mathcal{O}_X$ is locally free of finite rank. If X is a scheme and $p \in X$ we set $\operatorname{codim}_p X = \dim \mathcal{O}_{X,p}$ and we will denote by $X^{(1)} = \{p \in X \mid \operatorname{codim}_p X = 1\}$ the set of codimension 1 points of X.

If N is an abelian group we set $D(N) = \underline{\operatorname{Hom}}_{\text{groups}}(N, \mathbb{G}_m)$ for the diagonalizable group associated to it, while if $f : G \longrightarrow S$ is an affine group scheme we set $\mathcal{O}_S[G] = f_*\mathcal{O}_G$.

Moreover we will call a $\mathcal{O}_Y[G]$-comodule structure on a quasi-coherent sheaf $\mathcal{F}$ simply a G-comodule structure.

If $\mathcal{F}$ is a quasi-coherent sheaf on a scheme X, the expression $s \in \mathcal{F}$ will always mean $s \in \mathcal{F}(X) = \mathrm{H}^0(X, \mathcal{F})$. Moreover, we will denote by $V(s)$ the zero locus of s in X, that is, the closed subscheme associated with the sheaf of ideals $\mathrm{Ker}(\mathcal{O}_X \xrightarrow{s} \mathcal{F})$.

Given an element $f = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ and invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on a scheme we will use the notation

$$\underline{\mathcal{L}}^f = \bigotimes_i \mathcal{L}_i^{\otimes a_i}, \quad \mathrm{Sym}^* \underline{\mathcal{L}} = \mathrm{Sym}^*(\mathcal{L}_1, \ldots, \mathcal{L}_r) = \bigoplus_{g \in \mathbb{Z}^r} \underline{\mathcal{L}}^g.$$

Note also that, if for any i, we have $\mathcal{L}_i = \mathcal{O}$, then there is a canonical isomorphism $\underline{\mathcal{L}}^f \simeq \mathcal{O}$.

Given $\alpha \in \mathbb{N}$, we will use the following convention:

$$0^\alpha = \begin{cases} 1 & \alpha = 0, \\ 0 & \alpha > 0. \end{cases}$$

We denote by (sets) the category of sets.

The abbreviation 'fppf' stands for 'faithfully flat of finite presentation'.

Finally, if $\mathcal{X}$ is an algebraic stack, we denote by $|\mathcal{X}|$ its associated topological space.

2 *G*-covers

In this section, we will fix a base scheme S and a flat and finite group scheme G finitely presented over S. We will denote by $\mathscr{A}$ the regular representation of G, that is, $\mathscr{A} = \mathcal{O}_S[G]$ with the $\mathcal{O}_Y[G]$-comodule structure $\mu : \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathcal{O}_S[G]$ induced by the multiplication of G.

The aim of this section is to introduce the notion of a ramified Galois cover and prove that the associated stack is algebraic.

Definition 2.1. Given a scheme T over S, a *ramified Galois cover of group G*, or simply a *G-cover*, over it is a cover $X \xrightarrow{f} T$ together with an action of G_T on it such that there exists an fppf covering $\{U_i \longrightarrow T\}$ and isomorphisms of G-comodules

$$(f_* \mathcal{O}_X)_{|U_i} \simeq \mathscr{A}_{|U_i}.$$

We will call G-Cov(T) the groupoid of G-covers over T, where the arrows are the G-equivariant isomorphisms of schemes over T. □

The G-covers form a stack G-Cov over S. Moreover any G-torsor is a G-cover and more precisely we have:

Proposition 2.2. BG is an open substack of G-Cov. □

Proof. Given a scheme U over S and a G-cover $X = \operatorname{Spec} \mathscr{B}$ over U, X is a G-torsor if and only if the map $G \times X \longrightarrow X \times X$ is an isomorphism. This map is induced by a map $\mathscr{B} \otimes \mathscr{B} \xrightarrow{h} \mathscr{B} \otimes \mathcal{O}[G_U]$ and so the locus over which X is a G-torsor is given by the vanishing of $\operatorname{Coker} h$, which is an open subset. ■

Definition 2.3. The main component $\mathcal{Z}_G$ of G-Cov is the reduced closed substack induced by the closure of BG in G-Cov. □

In order to prove that G-Cov is an algebraic stack we will present it as a quotient stack by a smooth group scheme.

Notation 2.4. Let S be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf over it. We denote by $\mathrm{W}(\mathcal{F}) : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow (\mathrm{sets})$ the functor

$$\mathrm{W}(\mathcal{F})(U \xrightarrow{f} S) = \mathrm{H}^0(U, f^*\mathcal{F}).$$

Structures of G-comodule over $\mathcal{F}$ correspond to left actions of G on the functor $\mathrm{W}(\mathcal{F})$.

If $\mathcal{H}$ is another quasi-coherent sheaf over S with a structure of G-comodule, there is an induced left action on the functor $\underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$. We denote by $\underline{\mathrm{Hom}}^G(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$ (resp. $\underline{\mathrm{End}}^G \mathrm{W}(\mathcal{F})$, $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathcal{F})$) the subfunctor of $\underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$ (resp. $\underline{\mathrm{End}}\, \mathrm{W}(\mathcal{F})$, $\underline{\mathrm{Aut}}\, \mathrm{W}(\mathcal{F})$) given by the G-invariants elements, that are exactly the G-equivariant morphisms. When $\mathcal{F}$ is locally free of finite rank, there is a natural isomorphism

$$\mathrm{W}(\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{H})) \longrightarrow \underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{F}), \mathrm{W}(\mathcal{H}))$$

that induces a G-comodule structure on the sheaf $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{H})$. The subsheaf of G-invariants, which we will denote by $\underline{\mathrm{Hom}}^G(\mathcal{F}, \mathcal{H})$, coincides with the subsheaf of $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{H})$ of morphisms preserving the G-comodule structures. Finally, we set $\underline{\mathrm{End}}^G(\mathcal{F}) = \underline{\mathrm{Hom}}^G(\mathcal{F}, \mathcal{F})$. □

Remark 2.5. If $\mathcal{F}$ is a locally free sheaf of finite rank, then $\mathrm{W}(\mathcal{F})$ is smooth and affine. $\square$

Proposition 2.6. The functor

$$\begin{tikzcd}
(\mathrm{Sch}/S)^{\mathrm{op}} \arrow[r, "X_G"] & (\text{sets}) \\
T \arrow[r, mapsto] & \left\{ \begin{array}{c} \text{algebra structures on } \mathscr{A}_T \\ \text{in the category of } G\text{-comodules} \end{array} \right\}
\end{tikzcd}$$

is an affine scheme finitely presented over S. $\square$

Proof. Let T be a scheme over S. An element of $X_G(T)$ is given by maps

$$\mathscr{A}_T \otimes \mathscr{A}_T \xrightarrow{m} \mathscr{A}_T, \quad \mathcal{O}_T \xrightarrow{e} \mathscr{A}_T$$

for which $\mathscr{A}$ becomes a sheaf of algebras with multiplication m and identity $e(1)$ and such that μ is a homomorphism of algebras over $\mathcal{O}_T$. In particular, e has to be an isomorphism onto $\mathscr{A}^G = \mathcal{O}_T$. Therefore, we have an inclusion $X_G \subseteq \underline{\mathrm{Hom}}(\mathrm{W}(\mathscr{A} \otimes \mathscr{A}), \mathrm{W}(\mathscr{A})) \times \mathbb{G}_m$, which turns out to be a closed immersion, since locally, after we choose a basis of $\mathscr{A}$, the above conditions translate in the vanishing of certain polynomials. $\blacksquare$

Proposition 2.7. $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A})$ is a smooth group scheme finitely presented over S. $\square$

Proof. If T is an S-scheme, the morphisms

$$\begin{tikzcd}
\varepsilon \circ \phi & \phi, \arrow[l, mapsto] \\
\mathcal{O}_T[G]^\vee \arrow[r] & \underline{\mathrm{End}}^G(\mathscr{A} \otimes \mathcal{O}_T), \\
f \arrow[r, mapsto] & (f \otimes \mathrm{id}) \circ \Delta,
\end{tikzcd}$$

where Δ and ε are, respectively, the co-multiplication and the co-unit of $\mathcal{O}_T[G]$, are inverses of each other. Since

$$\mathrm{W}(\mathcal{O}_S[G]^\vee) \simeq \underline{\mathrm{Hom}}(\mathrm{W}(\mathcal{O}_S[G]), \mathrm{W}(\mathcal{O}_S)),$$

we obtain an isomorphism $\underline{\mathrm{End}}^G \mathrm{W}(\mathscr{A}) \simeq \mathrm{W}(\mathcal{O}_S[G]^\vee)$, so that $\underline{\mathrm{End}}^G \mathrm{W}(\mathscr{A})$ and its open subscheme $\underline{\mathrm{Aut}}^G \mathrm{W}(\mathscr{A})$ are smooth and finitely presented over S. $\blacksquare$

Remark 2.8. $\underline{\mathrm{Aut}}^G\mathrm{W}(\mathcal{A})$ acts on X_G in the following way. Given a scheme T over S, a G-equivariant automorphism $f:\mathcal{A}_T\longrightarrow\mathcal{A}_T$ and $(m,e)\in X_G(T)$ we can set $f(m,e)$ for the unique structure of sheaf of algebras on $\mathcal{A}_T$ such that $f:(\mathcal{A}_T,m,e)\longrightarrow(\mathcal{A}_T,f(m,e))$ is an isomorphism of $\mathcal{O}_T$-algebras. □

Proposition 2.9. The natural map $X_G\xrightarrow{\pi}G\text{-Cov}$ is an $\underline{\mathrm{Aut}}^G\mathrm{W}(\mathcal{A})$-torsor, that is,

$$G\text{-Cov}\simeq[X_G/\underline{\mathrm{Aut}}^G\mathrm{W}(\mathcal{A})].$$ □

Proof. Consider a cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X_G \arrow[d, "\pi"] \\ U \arrow[r, "f"'] & G\text{-Cov} \end{tikzcd}$$

where U is a scheme and $f:Y\longrightarrow U$ is a G-cover. We want to prove that P is an $\underline{\mathrm{Aut}}^G\mathrm{W}(\mathcal{A})$ torsor over U and that the map $P\longrightarrow X_G$ is equivariant. Since π is an fppf epimorphism, we can assume that f comes from X_G, that is, $f_*\mathcal{O}_Y=\mathcal{A}_U$ with multiplication m and neutral element e. It is now easy to prove that

$$\begin{array}{rcl}\underline{\mathrm{Aut}}^G\mathrm{W}(\mathcal{A}_U) & \xrightarrow{\simeq} & P,\\ h & \longmapsto & h(m,e)\end{array}$$

is a bijection and that all the other claims hold. ■

Using the above propositions, we can conclude that:

Theorem 2.10. The stack G-Cov is algebraic and finitely presented over S. □

3 The Stack $\mathcal{X}_\phi$

In the following sections, we will study the stack G-Cov when $G=\mathrm{D}(M)$, the diagonalizable group of a finite abelian group M. The structure of this stack and of some of its substacks is somehow special and in this section we will provide general constructions and properties that will be used later. To a monoid map $T_+\xrightarrow{\phi}\mathbb{Z}^r$, we will associate a stack $\mathcal{X}_\phi$ whose objects are sequences of invertible sheaves with additional data and we will study particular "parametrization" of these objects, defined by a map of stacks $\mathcal{F}_{\underline{\mathcal{E}}} \xrightarrow{\pi_{\underline{\mathcal{E}}}} \mathcal{X}_\phi$, where $\mathcal{F}_{\underline{\mathcal{E}}}$ will be a "nice" stack, for instance, smooth.

In this section, we will consider given a commutative monoid T_+ together to a monoid map $\phi : T_+ \longrightarrow \mathbb{Z}^r$.

Definition 3.1. We define the stack $\mathcal{X}_\phi$ over $\mathbb{Z}$ as follows.

- *Objects.* An object over a scheme S is a pair $(\underline{\mathcal{L}}, a)$ where:
    - $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_r$ are invertible sheaves on S;
    - $T_+ \xrightarrow{a} \mathrm{Sym}^* \underline{\mathcal{L}}$ is an additive map such that $a(t) \in \underline{\mathcal{L}}^{\phi(t)}$ for any $t \in T_+$.
- *Arrows.* An isomorphism $(\underline{\mathcal{L}}, a) \xrightarrow{\underline{\sigma}} (\underline{\mathcal{L}}', a')$ of objects over S is given by a sequence $\underline{\sigma} = \sigma_1, \dots, \sigma_r$ of isomorphisms $\sigma_i : \mathcal{L}_i \xrightarrow{\simeq} \mathcal{L}'_i$ such that

$$\underline{\sigma}^{\phi(t)}(a(t)) = a'(t) \quad \text{for any } t \in T_+. \qquad \square$$

Example 3.1. Let $f_1, \dots, f_s, g_1, \dots, g_t \in \mathbb{Z}^r$ and consider the stack $\mathcal{X}_{\underline{f},\underline{g}}$ of invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ with maps $\mathcal{O} \longrightarrow \underline{\mathcal{L}}^{f_i}$ and $\mathcal{O} \xrightarrow{\simeq} \underline{\mathcal{L}}^{g_j}$. If $T_+ = \mathbb{N}^s \times \mathbb{Z}^t$ and $\phi : T_+ \longrightarrow \mathbb{Z}^r$ is the map given by the matrix $(f_1|\cdots|f_s|g_1|\cdots|g_t)$ then $\mathcal{X}_{\underline{f},\underline{g}} = \mathcal{X}_\phi$. $\square$

Notation 3.2. We set

$$\mathbb{Z}[T_+] = \mathbb{Z}[x_t]_{t \in T_+} / (x_t x_{t'} - x_{t+t'}, x_0 - 1)$$

and $\mathcal{O}_S[T_+] = \mathbb{Z}[T_+] \otimes_{\mathbb{Z}} \mathcal{O}_S$. The scheme $\mathrm{Spec}\, \mathcal{O}_S[T_+]$ over S represents the functor that associates to any scheme U/S the set of additive maps $T_+ \longrightarrow (\mathcal{O}_U, \cdot)$, where $\cdot$ denotes the multiplication on $\mathcal{O}_U$. The group $\mathrm{D}(\mathbb{Z}^r)$ acts on $\mathrm{Spec}\, \mathbb{Z}[T_+]$ by the graduation $\deg x_t = \phi(t)$. $\square$

Proposition 3.3. Set $X = \mathrm{Spec}\, \mathbb{Z}[T_+]$. The choice $\mathcal{L}_i = \mathcal{O}_X$ and

$$\begin{tikzcd}
\underline{\mathcal{L}}^{\phi(t)} \arrow[r, "\simeq"] & \mathcal{O}_X \\
a(t) \arrow[r, leftrightarrow] & x_t
\end{tikzcd}$$

induces a smooth epimorphism $X \longrightarrow \mathcal{X}_\phi$ such that $\mathcal{X}_\phi \simeq [X/\mathrm{D}(\mathbb{Z}^r)]$. In particular, $\mathcal{X}_\phi$ is an algebraic stack. $\square$

Proof. It is enough to note that an object of $[X/\mathrm{D}(\mathbb{Z}^r)](U)$ is given by invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ with a $\mathrm{D}(\mathbb{Z}^r)$-equivariant map $\mathrm{Spec}\,\mathrm{Sym}^*\underline{\mathcal{L}} \longrightarrow \mathrm{Spec}\,\mathbb{Z}[T_+]$ which exactly corresponds to an additive map $T_+ \longrightarrow \mathrm{Sym}^*\underline{\mathcal{L}}$ as in the definition of $\mathcal{X}_\phi$. It is easy to check that the map $X \longrightarrow [X/\mathrm{D}(\mathbb{Z}^r)] \longrightarrow \mathcal{X}_\phi$ is the one defined in the statement. ■

Remark 3.4. Given a map $U \xrightarrow{a} X = \mathrm{Spec}\,\mathbb{Z}[T_+]$, that is, a monoid map $T_+ \xrightarrow{a} \mathcal{O}_U$, the induced object $U \xrightarrow{a} X \longrightarrow \mathcal{X}_\phi$ is the pair $(\underline{\mathcal{L}}, \tilde{a})$, where $\mathcal{L}_i = \mathcal{O}_U$ and for any $t \in T_+$

$$\begin{tikzcd}
\mathcal{O}_U \arrow[r, "\simeq"] & \underline{\mathcal{L}}^{\phi(t)} \\
a(t) \arrow[r, mapsto] & \tilde{a}(t)
\end{tikzcd}$$

We will denote by a also the object $(\underline{\mathcal{L}}, \tilde{a}) \in \mathcal{X}_\phi(U)$.

Given two elements $a, b : T_+ \longrightarrow \mathcal{O}_U \in \mathcal{X}_\phi(U)$, we have

$$\mathrm{Iso}_{\mathcal{X}_\phi(U)}(a, b) = \{\sigma_1, \dots, \sigma_r \in \mathcal{O}_U^* \mid \underline{\sigma}^{\phi(t)} a(t) = b(t)\ \forall\, t \in T_+\}.$$

□

Lemma 3.5. Consider a commutative diagram:

$$\begin{tikzcd}
T_+ \arrow[r, "h"] \arrow[d, "\phi"'] & T'_+ \arrow[d, "\psi"] \\
\mathbb{Z}^r \arrow[r, "g"] & \mathbb{Z}^s
\end{tikzcd}$$

where T_+ and T'_+ are commutative monoids and ϕ, ψ, h, and g are additive maps. Then we have a 2-commutative diagram:

$$\begin{tikzcd}
\mathrm{Spec}\,\mathbb{Z}[T'_+] \arrow[r, "h^*"] \arrow[d] & \mathrm{Spec}\,\mathbb{Z}[T_+] \arrow[d] \\
\mathcal{X}_\psi \arrow[r, "\Lambda"] & \mathcal{X}_\phi
\end{tikzcd}$$

$$(\underline{\mathcal{L}}, T'_+ \xrightarrow{a} \mathrm{Sym}^*\underline{\mathcal{L}}) \longmapsto (\underline{\mathcal{M}}, T_+ \xrightarrow{b} \mathrm{Sym}^*\underline{\mathcal{M}}) \tag{3.1}$$

where, for $i = 1, \dots, r$, $\mathcal{M}_i = \underline{\mathcal{L}}^{g(e_i)}$ and b is the unique map such that

$$\begin{tikzcd}
T_+ \arrow[r, "b"] \arrow[d, "h"'] & \mathrm{Sym}^*\underline{\mathcal{M}} \arrow[d] & \underline{\mathcal{M}}^v \arrow[d] \arrow[r, phantom, "\simeq"] & \underline{\mathcal{L}}^{g(v)} \arrow[dl, "\mathrm{id}"] \\
T'_+ \arrow[r, "a"'] & \mathrm{Sym}^*\underline{\mathcal{L}} & \underline{\mathcal{L}}^{g(v)} &
\end{tikzcd}$$

□

Proof. An easy computation shows that there is a canonical isomorphism $\underline{\mathcal{M}}^v \simeq \underline{\mathcal{L}}^{g(v)}$ for all $v \in \mathbb{Z}^r$ and so $b(t)$ corresponds under this isomorphism to $a(h(t)) \in \underline{\mathcal{L}}^{\psi(h(t))} = \underline{\mathcal{L}}^{g(\phi(t))} \simeq \underline{\mathcal{M}}^{\phi(t)}$. So the functor Λ is well defined and we only have to check the commutativity of the second diagram in the statement. The map $\mathrm{Spec}\,\mathbb{Z}[T'_+] \longrightarrow \mathrm{Spec}\,\mathbb{Z}[T_+] \longrightarrow \mathcal{X}_\phi$ is given by trivial invertible sheaves and the additive map

$$\begin{tikzcd}
T_+ \arrow[r] & \mathbb{Z}[T_+][x_1, \dots, x_r]_{\prod_i x_i} \arrow[r] & \mathbb{Z}[T'_+][x_1, \dots, x_r]_{\prod_i x_i} \\
t \arrow[r, mapsto] & x_t x^{\phi(t)} \arrow[r, mapsto] & x_{h(t)} x^{\phi(t)}
\end{tikzcd}$$

Instead the map $\mathrm{Spec}\,\mathbb{Z}[T'_+] \longrightarrow \mathcal{X}_\psi \longrightarrow \mathcal{X}_\phi$ is given by trivial invertible sheaves and the map b that makes the following diagram commutative:

$$\begin{tikzcd}
T_+ \arrow[r, "b"] \arrow[d, "h"'] & \mathbb{Z}[T'_+][x_1, \dots, x_r]_{\prod_i x_i} \arrow[d] & x^v \arrow[d, mapsto] \\
T'_+ \arrow[r, "a"] & \mathbb{Z}[T'_+][y_1, \dots, y_s]_{\prod_i y_i} & y^{g(v)} \\
t \arrow[r, mapsto] & x_t y^{\psi(t)} &
\end{tikzcd}$$

Since $x_{h(t)} x^{\phi(t)}$ is sent to $x_{h(t)} y^{g(\phi(t))} = x_{h(t)} y^{\psi(h(t))} = a(h(t))$ we find again $b(t) = x_{h(t)} x^{\phi(t)}$. ■

Remark 3.6. The functor $\mathcal{X}_\psi \longrightarrow \mathcal{X}_\phi$ sends an element $a : T'_+ \longrightarrow \mathcal{O}_U \in \mathcal{X}_\psi(U)$ to the element $a \circ h \in \mathcal{X}_\phi(U)$. Moreover, taking into account the description given in 3.4, if $a, b : T'_+ \longrightarrow \mathcal{O}_U \in \mathcal{X}_\psi(U)$, we have

$$\begin{tikzcd}
\mathrm{Iso}_U(a, b) \arrow[r] & \mathrm{Iso}_U(a \circ h, b \circ h) \\
\underline{\sigma} \arrow[r, mapsto] & \underline{\sigma}^{g(e_1)}, \dots, \underline{\sigma}^{g(e_r)}
\end{tikzcd}$$

□

3.1 The main irreducible component $\mathcal{Z}_\phi$ of $\mathcal{X}_\phi$

Notation 3.7. A monoid will be called *integral* if it satisfies the cancellation law, that is,

$$\forall\, a, b, c, \quad a+b=a+c \Longrightarrow b=c.$$

Let T_+ be a monoid. There exists, up to a unique isomorphism, a group T (resp. integral monoid T_+^{int}) such that any monoid map $T_+ \longrightarrow S_+$, where S_+ is a group (resp. integral monoid), factors uniquely through T (resp. T_+^{int}). We call it the *associated group* (resp. *associated integral monoid*) of T_+. Note that if T is the associated group of T_+, then $\text{Im}(T_+ \longrightarrow T)$ can be chosen as the associated integral monoid of T_+. We will continue to denote by T the associated group of T_+ and we set $T_+^{\text{int}} = \text{Im}(T_+ \longrightarrow T) \subseteq T$. In particular, $\langle T_+^{\text{int}} \rangle_{\mathbb{Z}} = T$.

From now on T_+ will be a finitely generated monoid whose associated group is a free $\mathbb{Z}$-module of finite rank. In order to simplify notation, we will often write $\phi : T \longrightarrow \mathbb{Z}^r$, meaning the extension of $\phi : T_+ \longrightarrow \mathbb{Z}^r$ to T. Anyway, the stack $\mathcal{X}_\phi$ will always be the stack $\mathcal{X}_{T_+ \longrightarrow \mathbb{Z}^r}$ and when we will have to consider the stack $\mathcal{X}_{T \longrightarrow \mathbb{Z}^r}$, we will always specify a different symbol for the induced map $T \longrightarrow \mathbb{Z}$. ☐

Remark 3.8. If D is a domain, then $\text{Spec}\, D[T]$ is an open subscheme of $\text{Spec}\, D[T_+]$, while $\text{Spec}\, D[T_+^{\text{int}}]$ is one of its irreducible components. In particular, we have the following: ☐

Proposition 3.9. Let $\hat{\phi} : T \longrightarrow \mathbb{Z}^r$ be the extension of ϕ and set $\phi^{\text{int}} = \hat{\phi}_{|T_+^{\text{int}}}$. Then $\mathcal{B}_\phi = \mathcal{X}_{\hat{\phi}} \longrightarrow \mathcal{X}_\phi$ is an open immersion, while $\mathcal{Z}_\phi = \mathcal{X}_{\phi^{\text{int}}} \longrightarrow \mathcal{X}_\phi$ is a closed one. Moreover, $\mathcal{Z}_\phi$ is the reduced closed stack associated to the closure of $\mathcal{B}_\phi$, it is an irreducible component of $\mathcal{X}_\phi$ and

$$\mathcal{B}_\phi \simeq [\text{Spec}\, \mathbb{Z}[T]/\text{D}(\mathbb{Z}^r)] \quad \text{and} \quad \mathcal{Z}_\phi \simeq [\text{Spec}\, \mathbb{Z}[T_+^{\text{int}}]/\text{D}(\mathbb{Z}^r)].$$ ☐

Definition 3.10. With notation above, we will call $\mathcal{B}_\phi$ and $\mathcal{Z}_\phi$ the *principal open substack* and the *main irreducible component* of $\mathcal{X}_\phi$, respectively. ☐

Notation 3.11. We set

$$T_+^\vee = \text{Hom}(T_+, \mathbb{N}) = \{\mathcal{E} \in \text{Hom}_{\text{groups}}(T, \mathbb{Z}) \mid \mathcal{E}(T_+) \subseteq \mathbb{N}\}.$$

We will call it the *dual monoid* of T_+ and we will call its elements the *rays* for T_+. Note that $T_+^\vee = T_+^{\text{int}^\vee}$. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$, we will denote by $\underline{\mathcal{E}}$ also the induced map $T \longrightarrow \mathbb{Z}^s$. Moreover we set

$$\operatorname{Supp} \underline{\mathcal{E}} = \{v \in T_+ \mid \exists i\, \mathcal{E}^i(v) > 0\}.$$

Finally, note that the dual monoid of a group is always 0. Therefore, when H is an abelian group, the dual $H^\vee$ of H will always be the dual as $\mathbb{Z}$-module. □

Definition 3.12. Given a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$ set

$$\begin{tikzcd} \mathbb{N}^s \oplus T \arrow[r, "\sigma_{\underline{\mathcal{E}}}"] & \mathbb{Z}^s \oplus \mathbb{Z}^r \\ (e_i, 0) \arrow[r, mapsto] & (e_i, 0) \\ (0, t) \arrow[r, mapsto] & (\underline{\mathcal{E}}(t), -\phi(t)) \end{tikzcd}$$

where $e_1, \dots, e_s$ is the canonical basis of $\mathbb{Z}^s$. We will call $\mathcal{F}_{\underline{\mathcal{E}}} = \mathcal{X}_{\sigma_{\underline{\mathcal{E}}}}$. □

Remark 3.13. An object of $\mathcal{F}_{\underline{\mathcal{E}}}$ over a scheme U is given by a sequence $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$ where

- $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_r$ and $\underline{\mathcal{M}} = (\mathcal{M}_{\mathcal{E}})_{\mathcal{E} \in \underline{\mathcal{E}}} = \mathcal{M}_1, \dots, \mathcal{M}_s$ are invertible sheaves on U;
- $\underline{z} = (z_{\mathcal{E}})_{\mathcal{E} \in \underline{\mathcal{E}}} = z_1, \dots, z_s$ are sections $z_i \in \mathcal{M}_i$;
- for any $t \in T$, $\lambda(t) = \lambda_t$ is an isomorphism $\underline{\mathcal{L}}^{\phi(t)} \xrightarrow{\simeq} \underline{\mathcal{M}}^{\underline{\mathcal{E}}(t)}$ additive in t.

An isomorphism $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \longrightarrow (\underline{\mathcal{L}}', \underline{\mathcal{M}}', \underline{z}', \lambda')$ is a pair $(\underline{\omega}, \underline{\tau})$ where $\underline{\omega} = \omega_1, \dots, \omega_r$, $\underline{\tau} = \tau_1, \dots, \tau_s$ are sequences of isomorphisms $\mathcal{L}_i \xrightarrow{\omega_i} \mathcal{L}_i'$, $\mathcal{M}_j \xrightarrow{\tau_j} \mathcal{M}_j'$ such that $\tau_j(z_j) = z_j'$ and for any $t \in T$ we have a commutative diagram:

$$\begin{tikzcd} \underline{\mathcal{L}}^{\phi(t)} \arrow[r, "\lambda_t"] \arrow[d, "\underline{\omega}^{\phi(t)}"'] & \underline{\mathcal{M}}^{\underline{\mathcal{E}}(t)} \arrow[d, "\underline{\tau}^{\phi(t)}"] \\ \underline{\mathcal{L}}'^{\phi(t)} \arrow[r, "\lambda_t'"] & \underline{\mathcal{M}}'^{\underline{\mathcal{E}}(t)} \end{tikzcd}$$

An object over U coming from the atlas $\operatorname{Spec} \mathbb{Z}[\mathbb{N}^s \oplus T]$ is a pair $(\underline{z}, \lambda)$ where $\underline{z} = z_1, \dots, z_s \in \mathcal{O}_U$ and $\lambda : T \longrightarrow \mathcal{O}_U^*$ is a group homomorphism. Given $(\underline{z}, \lambda), (\underline{z}', \lambda') \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$, we have

$$\operatorname{Iso}_U((\underline{z}, \lambda), (\underline{z}', \lambda')) = \{(\underline{\omega}, \underline{\tau}) \in (\mathcal{O}_U^*)^r \times (\mathcal{O}_U^*)^s \mid \tau_i z_i = z_i', \underline{\tau}^{\underline{\mathcal{E}}(t)} \lambda(t) = \underline{\omega}^{\phi(t)} \lambda'(t)\}.$$ □

Definition 3.14. Given a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s$ of elements of $T_+^\vee$, we define the map

$$\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi$$

induced by the commutative diagram

$$\begin{tikzcd}
t \arrow[d, mapsto] & T_+ \arrow[r, "\phi"] \arrow[d] & \mathbb{Z}^r \arrow[d, hook] \\
(\underline{\mathcal{E}}(t), -t) & \mathbb{N}^s \oplus T \arrow[r, "\sigma_{\underline{\mathcal{E}}}"'] & \mathbb{Z}^s \oplus \mathbb{Z}^r
\end{tikzcd}$$

□

Remark 3.15. We can describe the functor $\pi_{\underline{\mathcal{E}}}$ explicitly. So suppose that we have an object $\chi = (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$. We have $\pi_{\underline{\mathcal{E}}}(\chi) = (\underline{\mathcal{L}}, a) \in \mathcal{X}_\phi(U)$ where a is given, for any $t \in T_+$, by

$$\begin{tikzcd}
\underline{\mathcal{L}}^{\phi(t)} \arrow[r, "\lambda_t"] & \underline{\mathcal{M}}^{\mathcal{E}(t)} \\
a(t) \arrow[r, mapsto] & \underline{z}^{\mathcal{E}(t)}
\end{tikzcd}$$

Moreover, if $(\underline{\omega}, \underline{\tau})$ is an isomorphism in $\mathcal{F}_{\underline{\mathcal{E}}}$, then $\pi_{\underline{\mathcal{E}}}(\underline{\omega}, \underline{\tau}) = \underline{\omega}$.

If $(\underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(U)$ then $a = \pi_{\underline{\mathcal{E}}}(\underline{z}, \lambda) \in \mathcal{X}_\phi(U)$ is given by

$$\begin{tikzcd}
T_+ \arrow[r] & \mathcal{O}_U \\
t \arrow[r, mapsto] & \underline{z}^{\mathcal{E}(t)}/\lambda_t = z_1^{\mathcal{E}^1(t)} \cdots z_s^{\mathcal{E}^s(t)}/\lambda_t
\end{tikzcd}$$

□

Remark 3.16. If $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$ is a sequence of elements of $T_+^\vee$, $J \subseteq I$, and we set $\underline{\delta} = (\mathcal{E}^j)_{j \in J}$ we can define a map over $\mathcal{X}_\phi$ as

$$\begin{tikzcd}
\mathcal{F}_{\underline{\delta}} \arrow[r, "\rho"] & \mathcal{F}_{\underline{\mathcal{E}}} \\
(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \arrow[r, mapsto] & (\underline{\mathcal{L}}, \underline{\mathcal{M}}', \underline{z}', \lambda)
\end{tikzcd} \qquad \mathcal{M}'_i = \begin{cases} \mathcal{M}_i & i \in J, \\ \mathcal{O} & i \notin J, \end{cases} \qquad z'_i = \begin{cases} z_i & i \in J, \\ 1 & i \notin J. \end{cases}$$

In fact ρ comes from the monoid map $T \oplus \mathbb{N}^I \longrightarrow T \oplus \mathbb{N}^J$ induced by the projection. Moreover, ρ is an open immersion, whose image is the open substack of $\mathcal{F}_{\underline{\mathcal{E}}}$ of objects $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$ such that z_i generates $\mathcal{M}_i$ for all $i \notin J$. We will often consider $\mathcal{F}_{\underline{\delta}}$ as an open substack of $\mathcal{F}_{\underline{\mathcal{E}}}$. □

Definition 3.17. Given a sequence $\underline{\mathcal{E}}=\mathcal{E}^1,\dots,\mathcal{E}^s$ of elements of $T_+^\vee$, we define

$$T_+^{\underline{\mathcal{E}}}=T_+^{\mathcal{E}^1,\dots,\mathcal{E}^s}=\{v\in T \mid \forall\, i\ \mathcal{E}^i(v)\geq 0\}.$$

We also consider the case $s=0$, so that $T_+^{\underline{\mathcal{E}}}=T$. If we denote by $\hat{\phi}: T_+^{\underline{\mathcal{E}}}\longrightarrow \mathbb{Z}^r$ the extension of ϕ, we also define $\mathcal{X}_\phi^{\underline{\mathcal{E}}}=\mathcal{Z}_\phi^{\underline{\mathcal{E}}}=\mathcal{X}_{\hat{\phi}}$. □

Remark 3.18. Assume that we have a monoid map $T_+\longrightarrow T'_+$ (compatible with ϕ and ϕ') inducing an isomorphism on the associated groups. If $\underline{\mathcal{E}}=\mathcal{E}^1,\dots,\mathcal{E}^s\in T_+'^\vee\subseteq T_+^\vee$, then we have a 2-commutative diagram

$$\begin{tikzcd}
\mathcal{F}'_{\underline{\mathcal{E}}} \arrow[r, "\simeq"] \arrow[d, "\pi'_{\underline{\mathcal{E}}}"'] & \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\
\mathcal{X}_{\phi'} \arrow[r] & \mathcal{X}_\phi
\end{tikzcd}$$

where $\mathcal{F}'_{\underline{\mathcal{E}}}$ is the stack obtained from T'_+ with respect to $\underline{\mathcal{E}}$. □

Proposition 3.19. The map $\pi_{\underline{\mathcal{E}}}:\mathcal{F}_{\underline{\mathcal{E}}}\longrightarrow\mathcal{X}_\phi$ has a natural factorization

$$\mathcal{F}_{\underline{\mathcal{E}}}\longrightarrow\mathcal{X}_\phi^{\underline{\mathcal{E}}}\longrightarrow\mathcal{Z}_\phi\longrightarrow\mathcal{X}_\phi.$$ □

Proof. The factorization follows from Remark 3.18 taking monoid maps $T_+\longrightarrow T_+^{\text{int}}\longrightarrow T_+^{\underline{\mathcal{E}}}$. ■

Remark 3.20. This shows that $\pi_{\underline{\mathcal{E}}}$ has image in $\mathcal{Z}_\phi$.We will call with the same symbol $\pi_{\underline{\mathcal{E}}}$ the factorization $\mathcal{F}_{\underline{\mathcal{E}}}\longrightarrow\mathcal{Z}_\phi$. □

We now want to show how the rays of T_+ can be used to describe the objects of $\mathcal{Z}_\phi$ over a field. Using notation from Remark 3.4, the result is as follows:

Theorem 3.21. Let k be a field and $T_+\xrightarrow{a}k\in\mathcal{X}_\phi(k)$. Then $a\in\mathcal{Z}_\phi(k)$ if and only if there exists a group homomorphism $\lambda: T\longrightarrow\bar{k}^*$ and $\mathcal{E}\in T_+^\vee$ such that

$$a(t)=\lambda_t 0^{\mathcal{E}(t)}$$

In particular, if $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ generate $T_+^\vee \otimes \mathbb{Q}$ then $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}}(\bar{k}) \longrightarrow \mathcal{Z}_\phi(\bar{k})$ is essentially surjective and so $\pi_{\underline{\mathcal{E}}} : |\mathcal{F}_{\underline{\mathcal{E}}}| \longrightarrow |\mathcal{Z}_\phi|$ is surjective. Finally, if the map $\phi : T \longrightarrow \mathbb{Z}^r$ is injective, we have a one-to-one correspondence

$$\begin{tikzcd}
\mathcal{Z}_\phi(\bar{k})/\simeq \arrow[r, "\gamma"] & \{X \subseteq T_+ \mid X = \operatorname{Supp} \mathcal{E} \text{ for } \mathcal{E} \in T_+^\vee\} \\
a \arrow[r, mapsto] & \{a = 0\}
\end{tikzcd}$$

In particular, $|\mathcal{Z}_\phi| = (\mathcal{Z}_\phi(\bar{\mathbb{Q}})/\simeq) \bigsqcup [\bigsqcup_{\text{primes } p} (\mathcal{Z}_\phi(\bar{\mathbb{F}}_p)/\simeq)]$. □

Before proving this theorem, we need some preliminary results that will also be useful later.

Definition 3.22. If T_+ is integral, $\mathcal{E} \in T_+^\vee$ and k is a field, we define

$$p_{\mathcal{E}} = \bigoplus_{v \in T_+, \mathcal{E}(v) > 0} k x_v \subseteq k[T_+].$$

If $p \in \operatorname{Spec} k[T_+]$, we set $p^{\mathrm{om}} = \bigoplus_{x_v \in p} k x_v$. □

The suffix $(-)^{\mathrm{om}}$ here stays for "homogeneous", since, when $T_+ = \mathbb{N}^r$ and $k[T_+] = k[x_1, \dots, x_r]$, p^{om} is an homogeneous ideal, actually a monomial ideal.

Lemma 3.23. Let k be a field and assume that T_+ is integral. Then:

(1) if $\mathcal{E} \in T_+^\vee$, $p_{\mathcal{E}}$ is prime and $k[\{v \in T_+ \mid \mathcal{E}(v) = 0\}] \longrightarrow k[T_+] \longrightarrow k[T_+]/p_{\mathcal{E}}$ is an isomorphism.

(2) If $p \in \operatorname{Spec} k[T_+]$ then $p^{\mathrm{om}} = p_{\mathcal{E}}$ for some $\mathcal{E} \in T_+^\vee$. □

Proof. (1) It is obvious.

(2) p^{om} is a prime thanks to [11, Proposition 1.7.12] and therefore $p^{\mathrm{om}} = p_{\mathcal{E}}$ for some $\mathcal{E} \in T_+^\vee$ thanks to [17, Corollary 2.2.4]. ■

Remark 3.24. If k is an algebraically closed field, $\phi : T \longrightarrow \mathbb{Z}^r$ is injective and $a, b \in \mathcal{X}_\phi(k)$ differ by a torsor, that is, there exists $\lambda : T_+ \longrightarrow k^*$ such that $a = \lambda b$, then $a \simeq b$ in $\mathcal{Z}_\phi(k)$. Indeed λ extends to a map $T \longrightarrow k^*$ and, since k is algebraically closed, it extends again to a map $\lambda : \mathbb{Z}^r \longrightarrow k^*$. □

Proof of Theorem 3.21. We can assume that k is algebraically closed and that T_+ is integral, since if a has an expression as in the statement then clearly $a \in \mathcal{Z}_\phi(k)$. Consider $p = \operatorname{Ker}(k[T_+] \xrightarrow{a} k)$. Thanks to Lemma 3.23, we can write $p^{\mathrm{om}} = p_{\mathcal{E}}$ for some $\mathcal{E} \in T_+^\vee$. Set $T'_+ = \{v \in T_+ \mid \mathcal{E}(v) = 0\}$ and $T' = \langle T'_+ \rangle_{\mathbb{Z}}$. Since a maps T'_+ to k^*, there exists an extension $\lambda : T' \longrightarrow k^*$. On the other hand, since k is algebraically closed, the inclusion $T' \longrightarrow T$ yields a surjection

$$\operatorname{Hom}(T, k^*) \longrightarrow \operatorname{Hom}(T', k^*)$$

and so we can extend again to an element $\lambda : T \longrightarrow k^*$. Since one has $\operatorname{Supp} \mathcal{E} = \{a = 0\}$ by construction, it is easy to check that $a(t) = \lambda_t 0^{\mathcal{E}(t)}$ for all $t \in T_+$.

Now consider the last part of the statement and so assume $\phi : T \longrightarrow \mathbb{Z}^r$ injective. The description of the objects $a \in \mathcal{Z}_\phi(k)$ given above shows that the map γ is well defined. Moreover, it is surjective because given $\mathcal{E} \in T_+^\vee$, one can always define $a(t) = 0^{\mathcal{E}(t)}$. For the injectivity, let $a, b \in \mathcal{Z}_\phi(k)$ be such that $\{a = 0\} = \{b = 0\}$. We can write $a(t) = \lambda_t 0^{\mathcal{E}(t)}$, $b(t) = \mu_t 0^{\mathcal{E}(t)}$, where $\lambda, \mu : T \longrightarrow k^*$, so that a and b differ by a torsor and are therefore isomorphic thanks to Remark 3.24. Finally, since any point of $|\mathcal{Z}_\phi|$ comes from an object of $\mathcal{Z}_\phi(\mathbb{Z})$, we also have the last equality. ■

In some cases, the description of the objects of $\mathcal{F}_{\underline{\mathcal{E}}}$ can be simplified, regardless of $\underline{\mathcal{E}}$, in the sense that there exist a stack of reduced data $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$, whose objects can be described by less data, and an isomorphism $\mathcal{F}_{\underline{\mathcal{E}}} \simeq \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$. This kind of simplification could be very useful when we have to deal with an explicit map of monoids $\phi : T_+ \longrightarrow \mathbb{Z}^r$, as we will see in Proposition 4.7. The idea is that in order to define an object $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}$, we do not really need all the invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$, because they are uniquely determined by a subset of them and the other data.

Definition 3.25. Assume $T \xrightarrow{\phi} \mathbb{Z}^r$ injective. Let $V \subseteq \mathbb{Z}^r$ be a submodule with a given basis $v_1, \dots, v_q$ and $\sigma : \mathbb{Z}^r \longrightarrow V$ be a map such that $(\mathrm{id} - \sigma)\mathbb{Z}^r \subseteq T$ (or equivalently $\pi = \pi \circ \sigma$ where π is the projection $\mathbb{Z}^r \longrightarrow \operatorname{Coker}\phi$). Define $W = \langle (\mathrm{id} - \sigma)V, \sigma T \rangle \subseteq V$. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^l \in T_+^\vee$ consider the map

$$\begin{array}{ccc} W \oplus \mathbb{N}^l & \xrightarrow{\psi_{\underline{\mathcal{E}},\sigma}} & \mathbb{Z}^q \oplus \mathbb{Z}^l \\ (w, z) & \longmapsto & (-w, \underline{\mathcal{E}}(w) + z) \end{array}$$

We define $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} = \mathcal{X}_{\psi_{\underline{\mathcal{E}},\sigma}}$ and we call it the stack of reduced data of $\underline{\mathcal{E}}$. □

Lemma 3.26. Consider a submodule $U \subseteq \mathbb{Z}^p$, a map $\underline{\mathcal{E}} : U \longrightarrow \mathbb{Z}^l$ and $\tau : \mathbb{Z}^p \longrightarrow \mathbb{Z}^p$ such that $(\mathrm{id} - \tau)\mathbb{Z}^p \subseteq U$. Consider the commutative diagram:

$$\begin{tikzcd}
(u, z) \arrow[d, mapsto] & U \oplus \mathbb{N}^l \arrow[r, "\tau\oplus\mathrm{id}"] \arrow[d, "\psi"'] & U \oplus \mathbb{N}^l \arrow[d, "\psi"] \\
(-u, \underline{\mathcal{E}}(u) + z) & \mathbb{Z}^p \oplus \mathbb{Z}^l \arrow[r] & \mathbb{Z}^p \oplus \mathbb{Z}^l \\
 & (u, z) \arrow[r, mapsto] & (\tau u, \underline{\mathcal{E}}(u - \tau u) + z)
\end{tikzcd}$$

Then the induced map $\varphi : \mathcal{X}_\psi \longrightarrow \mathcal{X}_\psi$ is isomorphic to $\mathrm{id}_{\mathcal{X}_\psi}$. □

Proof. Let $x_1, \dots, x_p$ be a $\mathbb{Z}$-basis of $\mathbb{Z}^p$ with $a_1, \dots, a_k \in \mathbb{N}$ such that $a_1 x_1, \dots, a_k x_k$ is a $\mathbb{Z}$-basis of U. We want to define a natural isomorphism $\mathrm{id}_{\mathcal{X}_\psi} \xrightarrow{\omega} \varphi$. First note that it is enough to define it on the objects of $\mathcal{X}_\psi$ coming from the atlas $\mathrm{Spec}\,\mathbb{Z}[U \oplus \mathbb{N}^l]$ and prove the naturality between such objects on a fixed scheme T and for the restrictions. An object coming from the atlas is of the form $(\lambda, \underline{z})$, where $\lambda : U \longrightarrow \mathcal{O}_T^*$ is an additive map and $\underline{z} = z_1, \dots, z_l \in \mathcal{O}_T$. Moreover, $\varphi(\lambda, \underline{z}) = (\tilde{\lambda}, \underline{z})$, where $\tilde{\lambda} = \lambda \circ \tau$. Let $\underline{\eta} \in \mathrm{D}(\mathbb{Z}^p)(T)$ the only elements such that $\underline{\eta}^{x_i} = \lambda(x_i - \tau x_i)$ for $i = 1, \dots, p$. These objects are well defined since $(\mathrm{id} - \tau)\mathbb{Z}^p \subseteq U$. We claim that $\omega_{T,(\lambda,\underline{z})} = (\underline{\eta}, \underline{1})$ is an isomorphism $(\lambda, \underline{z}) \longrightarrow \varphi(\lambda, \underline{z})$ and define a natural transformation. It is an isomorphism since $1 z_j = z_j$ and the condition

$$\underline{\eta}^{-u} \underline{1}^{\underline{\mathcal{E}}(u)} \lambda(u) = \lambda(\tau u) \quad \forall\, u \in U$$

holds by construction checking it on the basis $a_1 x_1, \dots, a_k x_k$ of U (see Remark 3.4). It is also easy to check that this isomorphisms commute with the change of basis. So it remains to prove that, if $(\underline{\sigma}, \underline{\mu})$ is an isomorphism $(\lambda, \underline{z}) \longrightarrow (\lambda', \underline{z}')$ then we have a commutative diagram:

$$\begin{tikzcd}
(\lambda, \underline{z}) \arrow[r, "(\underline{\sigma}, \underline{\mu})"] \arrow[d, "\omega_{T,(\lambda,\underline{z})}"'] & (\lambda', \underline{z}') \arrow[d, "\omega_{T,(\lambda',\underline{z}')}"] \\
\varphi(\lambda, \underline{z}) \arrow[r, "\varphi(\underline{\sigma}, \underline{\mu})"] & \varphi(\lambda', \underline{z}')
\end{tikzcd}$$

We have $\varphi(\underline{\sigma}, \underline{\mu}) = (\underline{\tilde{\sigma}}, \underline{\tilde{\mu}})$ with $\underline{\tilde{\mu}} = \underline{\mu}$ and $\underline{\tilde{\sigma}}^{x_i} = \underline{\sigma}^{\tau x_i} \underline{\mu}^{\underline{\mathcal{E}}(x_i - \tau x_i)}$ (see Remark 3.6). So it is easy to check that the commutativity in the second member holds. For the first, the condition is $\tilde{\underline{\sigma}}\underline{\eta} = \underline{\eta}'\underline{\sigma}$, which is equivalent to

$$(\tilde{\underline{\sigma}}\underline{\eta})^{x_i} = \underline{\sigma}^{\tau x_i}\underline{\mu}^{\underline{\mathcal{E}}(x_i - \tau x_i)}\lambda(x_i - \tau x_i) = (\underline{\eta}'\underline{\sigma})^{x_i} = \lambda'(x_i - \tau x_i)\underline{\sigma}^{x_i}$$

and to $\underline{\sigma}^{-(x_i - \tau x_i)}\underline{\mu}^{\underline{\mathcal{E}}(x_i - \tau x_i)}\lambda(x_i - \tau x_i) = \lambda'(x_i - \tau x_i)$ for any i. But, since $(\underline{\sigma}, \underline{\mu})$ is an isomorphism $(\lambda, \underline{z}) \longrightarrow (\lambda', \underline{z}')$, the condition

$$\underline{\sigma}^{-u}\underline{\mu}^{\underline{\mathcal{E}}(u)}\lambda(u) = \lambda'(u) \quad \forall\, u \in U$$

has to be satisfied. ■

Proposition 3.27. Assume $T \xrightarrow{\phi} \mathbb{Z}^r$ injective and let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T_+^\vee$ and $\sigma, V, v_1, \dots, v_q$ be as in Definition 3.25. For appropriate choices of isomorphisms $\tilde{\lambda}$ given by Lemma 3.5, the functors

$$\begin{tikzcd}
((\underline{\mathcal{N}}^{\sigma e_i} \otimes \underline{\mathcal{M}}^{\underline{\mathcal{E}}(e_i - \sigma e_i)})_{i=1,\dots,r}, \underline{\mathcal{M}}, \underline{z}, \tilde{\lambda}) & (\underline{\mathcal{N}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \arrow[l, mapsto] \\
\mathcal{F}_{\underline{\mathcal{E}}} \arrow[r, leftrightarrow] & \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} \\
(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \arrow[r, mapsto] & ((\underline{\mathcal{L}}^{v_i})_{i=1,\dots,q}, \underline{\mathcal{M}}, \underline{z}, \lambda_{|W})
\end{tikzcd}$$

are inverses of each other. □

Proof. Consider the commutative diagrams:

$$\begin{tikzcd}
W \oplus \mathbb{N}^s \arrow[r, hook] \arrow[d, "\psi"'] & T \oplus \mathbb{N}^s \arrow[d, "\phi_{\underline{\mathcal{E}}}"] \\
\mathbb{Z}^q \oplus \mathbb{Z}^s \arrow[r, hook] & \mathbb{Z}^r \oplus \mathbb{Z}^s
\end{tikzcd}
\qquad
\begin{tikzcd}
T \oplus \mathbb{N}^s \arrow[r, "\sigma \oplus \mathrm{id}"] \arrow[d, "\phi_{\underline{\mathcal{E}}}"'] & W \oplus \mathbb{N}^s \arrow[d, "\psi"] \\
\mathbb{Z}^r \oplus \mathbb{Z}^s \arrow[r] & \mathbb{Z}^q \oplus \mathbb{Z}^s \\
(x, y) \arrow[r, mapsto] & (\sigma x, \underline{\mathcal{E}}(x - \sigma x) + y)
\end{tikzcd}$$

They induce functors $\Lambda : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma}$ and $\Delta : \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}$, respectively, that behave as the functors of the statement thanks to the description given in Lemma 3.5. Finally, applying Lemma 3.26, we obtain that $\Lambda \circ \Delta \simeq \mathrm{id}$ and $\Delta \circ \Lambda \simeq \mathrm{id}$. ■

3.2 Extremal rays and smooth sequences

We continue to use notation from Notation 3.7. We have seen that given a collection $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T_+^\vee$ we can associate to it a stack $\mathcal{F}_{\underline{\mathcal{E}}}$ and a "parametrization" map $\mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{X}_\phi$. The stack $\mathcal{F}_{\underline{\mathcal{E}}}$ could be "too big" if we do not make an appropriate choice of the collection $\underline{\mathcal{E}}$. This happens for example if the rays in $\underline{\mathcal{E}}$ are not distinct or, more generally, if a ray in $\underline{\mathcal{E}}$ belongs to the submonoid generated by the other rays in $\underline{\mathcal{E}}$. Thus, we want to restrict our attention to a special class of rays, called extremal and to special sequences of them.

Definition 3.28. An *extremal* ray for T_+ is an element $\mathcal{E} \in T_+^\vee$ such that

- $\mathcal{E}$ has minimal nonempty support, that is, the set $\operatorname{Supp} \mathcal{E} \subseteq T_+$ is minimal in
$$(\{X \subseteq T_+ \mid X \neq \emptyset \text{ and } X = \operatorname{Supp} \delta \text{ for some } \delta \in T_+^\vee\}, \subseteq).$$
- $\mathcal{E}$ is normalized, that is, $\mathcal{E} : T \longrightarrow \mathbb{Z}$ is surjective. □

Lemma 3.29. Assume that T_+ is an integral monoid and let $v_1, \dots, v_l$ be a system of generators of T_+. Then the extremal rays are the normalized $\mathcal{E} \in T_+^\vee - \{0\}$ such that $\operatorname{Ker} \mathcal{E}$ contains $\operatorname{rk} T - 1$ $\mathbb{Q}$-independent vectors among the $v_1, \dots, v_l$. In particular, they are finitely many and they generate $\mathbb{Q}_+ T_+^\vee$. □

Proof. Denote by $\Omega \subseteq T_+^\vee$, the set of elements defined in the statement. From [7, Section 1.2, (9)], it follows that $\mathbb{Q}_+\Omega = \mathbb{Q}_+ T_+^\vee$. If $\mathcal{E} \in \Omega$ then it is an extremal ray. Indeed,

$$\emptyset \neq \operatorname{Supp} \mathcal{E}' \subseteq \operatorname{Supp} \mathcal{E} \Longrightarrow \exists \lambda \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E}' = \lambda \mathcal{E} \Longrightarrow \operatorname{Supp} \mathcal{E}' = \operatorname{Supp} \mathcal{E}.$$

Conversely, let $\mathcal{E}$ be an extremal ray and consider an expression

$$\mathcal{E} = \sum_{\delta \in \Omega} \lambda_\delta \delta \quad \text{with } \lambda_\delta \in \mathbb{Q}_{\geq 0}.$$

There must exists δ such that $\lambda_\delta \neq 0$. So

$$\operatorname{Supp} \delta \subseteq \operatorname{Supp} \mathcal{E} \Longrightarrow \operatorname{Supp} \delta = \operatorname{Supp} \mathcal{E} \Longrightarrow \exists \mu \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E} = \mu \delta \Longrightarrow \mathcal{E} = \delta.$$ ■

Corollary 3.30. For an extremal ray $\mathcal{E}$ and $\mathcal{E}' \in T_+^\vee$. we have

$$\operatorname{Supp}\mathcal{E}' = \operatorname{Supp}\mathcal{E} \iff \exists\lambda \in \mathbb{Q}_+ \text{ s.t. } \mathcal{E}' = \lambda\mathcal{E} \iff \exists\lambda \in \mathbb{N}_+ \text{ s.t. } \mathcal{E}' = \lambda\mathcal{E}. \qquad \square$$

Definition 3.31. An element $v \in T_+$ is said *indecomposable* if whenever $v = v' + v''$ with $v', v'' \in T_+$ it follows that $v' = 0$ or $v'' = 0$. $\square$

Proposition 3.32. $T_+^\vee$ has a unique minimal system of generators composed by the indecomposable elements. Moreover, any extremal ray is indecomposable. $\square$

Proof. The first claim of the statement follows from [17, Proposition 2.1.2] since $T_+^\vee$ is sharp, that is, it does not contain invertible elements. For the second, consider an extremal ray $\mathcal{E}$ and assume $\mathcal{E} = \mathcal{E}' + \mathcal{E}''$. We have

$$\operatorname{Supp}\mathcal{E}', \operatorname{Supp}\mathcal{E}'' \subseteq \operatorname{Supp}\mathcal{E} \Longrightarrow \mathcal{E}' = \lambda\mathcal{E}, \mathcal{E}'' = \mu\mathcal{E} \quad \text{with } \lambda, \mu \in \mathbb{N}$$

and so $\mathcal{E} = (\lambda + \mu)\mathcal{E} \Longrightarrow \lambda + \mu = 1 \Longrightarrow \lambda = 0$ or $\mu = 0 \Longrightarrow \mathcal{E}' = 0$ or $\mathcal{E}'' = 0$. ■

Definition 3.33. A *smooth sequence* for T_+ is a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^s \in T_+^\vee$ for which there exist elements $v_1, \dots, v_s$ in the associated integral monoid T_+^{int} of T_+ such that

$$T_+^{\text{int}} \cap \operatorname{Ker}\underline{\mathcal{E}} \text{ generates } \operatorname{Ker}\underline{\mathcal{E}} \quad \text{and} \quad \mathcal{E}^i(v_j) = \delta_{i,j} \quad \text{for all } i, j.$$

We will also say that a ray $\mathcal{E} \in T_+^\vee - \{0\}$ is *smooth* if there exists a smooth sequence as above such that $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^s \rangle_{\mathbb{N}}$ or, equivalently, such that $\operatorname{Supp}\mathcal{E} \subseteq \operatorname{Supp}\underline{\mathcal{E}}$. $\square$

Remark 3.34. If T_+ is integral and Ω is a system of generators, one can always assume that $v_i \in \Omega$. Moreover, we also have that $\Omega \cap \operatorname{Ker}\underline{\mathcal{E}}$ generates $\operatorname{Ker}\underline{\mathcal{E}}$.

Finally, the equivalence in the last sentence of Definition 3.33 follows from the fact that, since $\operatorname{Ker}\underline{\mathcal{E}}$ is generated by elements in T_+^{int}, then the inclusion of the supports implies that $\mathcal{E}_{|\operatorname{Ker}\underline{\mathcal{E}}} = 0$ and therefore $\mathcal{E} = \sum_i \mathcal{E}(v_i)\mathcal{E}^i$. $\square$

Lemma 3.35. Let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ be a smooth sequence. Then

$$T_+^{\underline{\mathcal{E}}} = \operatorname{Ker}\underline{\mathcal{E}} \oplus \langle v_1, \dots, v_r \rangle_{\mathbb{N}} \subseteq T \quad \text{where } v_1, \dots, v_r \in T_+^{\text{int}}, \mathcal{E}^i(v_j) = \delta_{i,j}.$$

Moreover, if $z_1, \dots, z_s \in T_+^{\mathrm{int}}$ generate T_+^{int}, then $\mathbb{Z}[T_+^{\underline{\mathcal{E}}}] = \mathbb{Z}[T_+^{\mathrm{int}}]_{\prod_{\underline{\mathcal{E}}(z_i)=0} x_{z_i}}$ so that $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$ ($\mathcal{X}_\phi^{\underline{\mathcal{E}}}$) is a smooth open subscheme (substack) of $\operatorname{Spec}\mathbb{Z}[T_+^{\mathrm{int}}]$ ($\mathcal{Z}_\phi$). □

Proof. We have $T = \operatorname{Ker}\underline{\mathcal{E}} \oplus \langle v_1, \dots, v_r\rangle_{\mathbb{Z}}$ and clearly $\operatorname{Ker}\underline{\mathcal{E}} \oplus \langle v_1, \dots, v_q\rangle_{\mathbb{N}} \subseteq T_+^{\underline{\mathcal{E}}}$. Conversely, if $v \in T_+^{\underline{\mathcal{E}}}$, we can write

$$v = z + \sum_i \mathcal{E}^i(v) v_i \quad \text{with } z \in \operatorname{Ker}\underline{\mathcal{E}} \Longrightarrow v \in \operatorname{Ker}\underline{\mathcal{E}} \oplus \langle v_1, \dots, v_q\rangle_{\mathbb{N}}.$$

In particular, $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\mathcal{E}}}] \simeq \mathbb{A}_{\mathbb{Z}}^r \times D_{\mathbb{Z}}(\operatorname{Ker}\underline{\mathcal{E}})$ and so both $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$ and $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$ are smooth. Now let

$$I = \{i \mid \underline{\mathcal{E}}(z_i) = 0\} \quad \text{and} \quad S_+ = \langle T_+^{\mathrm{int}}, -z_i \text{ for } i \in I\rangle \subseteq T.$$

We need to prove that $S_+ = T_+^{\underline{\mathcal{E}}}$. Clearly, we have the inclusion $\subseteq$. For the reverse inclusion, it is enough to prove that $-\operatorname{Ker}\underline{\mathcal{E}} \cap T_+^{\mathrm{int}} \subseteq S_+$. But if $v \in \operatorname{Ker}\underline{\mathcal{E}} \cap T_+^{\mathrm{int}}$ then

$$v = \sum_{j=1}^{s} a_j z_j = \sum_{j \in I} a_j z_j \Longrightarrow -v \in S_+.$$

■

Remark 3.36. Any subsequence of a smooth sequence is smooth too. Indeed, let $\underline{\delta} = \mathcal{E}^1, \dots, \mathcal{E}^s$ be a subsequence of a smooth sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$, with $r > s$. We have to prove that $\langle \operatorname{Ker}\underline{\delta} \cap T_+^{\mathrm{int}}\rangle_{\mathbb{Z}} = \operatorname{Ker}\underline{\delta}$. Take $v \in \operatorname{Ker}\underline{\delta}$. So

$$v - \sum_{j=s+1}^{r} \mathcal{E}^j(v) v_j \in \operatorname{Ker}\underline{\mathcal{E}} = \langle \operatorname{Ker}\underline{\mathcal{E}} \cap T_+^{\mathrm{int}}\rangle_{\mathbb{Z}} \subseteq \langle \operatorname{Ker}\underline{\delta} \cap T_+^{\mathrm{int}}\rangle_{\mathbb{Z}} \Longrightarrow v \in \langle \operatorname{Ker}\underline{\delta} \cap T_+^{\mathrm{int}}\rangle_{\mathbb{Z}}.$$

□

Proposition 3.37. Let $\mathcal{E} \in T_+^{\vee}$. Then $\mathcal{E}$ is a smooth extremal ray if and only if $\mathcal{E}$ is a smooth sequence composed of one element, that is, $\operatorname{Ker}\mathcal{E} \cap T_+^{\mathrm{int}}$ generates $\operatorname{Ker}\mathcal{E}$ and there exists $v \in T_+$ such that $\mathcal{E}(v) = 1$.

In particular, any element of a smooth sequence is a smooth extremal ray. □

Proof. We can assume that T_+ is integral. If $\mathcal{E}$ is smooth and extremal, then there exists a smooth sequence $\mathcal{E}^1, \dots, \mathcal{E}^q$ such that $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q\rangle_{\mathbb{N}}$. Since $\mathcal{E}$ is indecomposable, it follows that $\mathcal{E} = \mathcal{E}^i$ for some i. Conversely, assume that $\mathcal{E}$ is a smooth sequence. So it is smooth by definition and it is normalized since $\mathcal{E}(v) = 1$ for some v. Finally, an inclusion $\operatorname{Supp}\delta \subseteq \operatorname{Supp}\mathcal{E}$ for $\delta \in T_+^{\vee}$ means that $\delta \in \langle \mathcal{E}\rangle_{\mathbb{N}}$, as remarked in Remark 3.34, and so $\operatorname{Supp}\delta = \emptyset$ or $\operatorname{Supp}\delta = \operatorname{Supp}\mathcal{E}$. ■

We conclude with a lemma that will be useful later.

Lemma 3.38. Let T_+ and T'_+ be integral monoids and $h\colon T \longrightarrow T'$ be a homomorphism such that $h(T_+) = T'_+$ and $\operatorname{Ker} h = \langle \operatorname{Ker} h \cap T_+ \rangle$. If $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in T'^{\vee}_+$, then

$$\underline{\mathcal{E}} \text{ smooth sequence for } T'_+ \iff \underline{\mathcal{E}} \circ h \text{ smooth sequence for } T_+. \qquad \square$$

Proof. Clearly, there exist $v_i \in T'_+$ such that $\mathcal{E}^i(v_j) = \delta_{i,j}$ if and only if there exist $w_i \in T_+$ such that $\mathcal{E}^i \circ h(w_j) = \delta_{i,j}$. On the other hand, we have a surjective morphism

$$\operatorname{Ker} \underline{\mathcal{E}} \circ h / \langle \operatorname{Ker} \underline{\mathcal{E}} \circ h \cap T_+ \rangle_{\mathbb{Z}} \longrightarrow \operatorname{Ker} \underline{\mathcal{E}} / \langle \operatorname{Ker} \underline{\mathcal{E}} \cap T'_+ \rangle_{\mathbb{Z}}.$$

In order to conclude it is enough to prove that this map is injective. So let $v \in T$ such that

$$h(v) = \sum_j a_j z_j \quad \text{with } a_j \in \mathbb{Z}, z_j \in T'_+, \underline{\mathcal{E}}(z_j) = 0.$$

Since $h(T_+) = T'_+$, there exists $y_j \in T_+$ such that $h(y_j) = z_j$. In particular, $y = \sum_j a_j y_j \in \langle \operatorname{Ker} \underline{\mathcal{E}} \circ h \cap T_+ \rangle_{\mathbb{Z}}$ and

$$v - y \in \operatorname{Ker} h = \langle \operatorname{Ker} h \cap T_+ \rangle \subseteq \langle \operatorname{Ker} \underline{\mathcal{E}} \circ h \cap T_+ \rangle. \qquad \blacksquare$$

3.3 The smooth locus $\mathcal{Z}_\phi^{\mathrm{sm}}$ of the main component $\mathcal{Z}_\phi$

Lemma 3.39. Let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^q$ be a smooth sequence and χ be a finite sequence of elements of $T_+^{\vee}$. Assume that all the elements of χ are distinct, each $\mathcal{E}^i$ is an element of χ and that for any δ in χ, we have

$$\delta \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}} \Longrightarrow \exists i \; \delta = \mathcal{E}^i.$$

As usual denote by π_χ the map $\mathcal{F}_\chi \longrightarrow \mathcal{X}_\phi$. Then we have an equivalence

$$\mathcal{F}_{\underline{\mathcal{E}}} = \pi_\chi^{-1}(\mathcal{X}_\phi^{\underline{\mathcal{E}}}) \xrightarrow{\simeq} \mathcal{X}_\phi^{\underline{\mathcal{E}}}. \qquad \square$$

Proof. Set $\chi = \mathcal{E}^1, \dots, \mathcal{E}^q, \eta^1, \dots, \eta^l = \underline{\mathcal{E}}, \underline{\eta}$. We first prove that $\pi_\chi^{-1}(\mathcal{X}_\phi^{\underline{\mathcal{E}}}) \subseteq \mathcal{F}_{\underline{\mathcal{E}}}$. Since they are open substacks, we can check this over an algebraically closed field k. Let $(\underline{z}, \lambda) \in \pi_\chi^{-1}(\mathcal{X}_\phi^{\underline{\mathcal{E}}})$ so that $a = \pi_\chi(\underline{z}, \lambda) = \underline{z}^{\underline{\mathcal{E}}}/\lambda : T_+ \longrightarrow k$ by Remark 3.15. We have to prove that $z_{\eta_j} \neq 0$.

Assume by contradiction that $z_{\eta_j} = 0$. Since we can write $a = b0^{\eta_j}$ and since a extends to $T_+^{\underline{\mathcal{E}}}$ so that $a(t) \neq 0$ if $t \in T_+ \cap \operatorname{Ker} \underline{\mathcal{E}}$, we have that η_j is 0 on $T_+ \cap \operatorname{Ker} \underline{\mathcal{E}}$. In particular,

$$\operatorname{Supp} \eta^j \subseteq \operatorname{Supp} \underline{\mathcal{E}} \Longrightarrow \eta^j \in \langle \mathcal{E}^1, \dots, \mathcal{E}^q \rangle_{\mathbb{N}} \Longrightarrow \exists i \; \eta^j = \mathcal{E}^i.$$

Thanks to Remark 3.16, it is enough to prove that if $\underline{\mathcal{E}}$ is a smooth sequence such that $T_+ = T_+^{\underline{\mathcal{E}}}$ then $\pi_{\underline{\mathcal{E}}}$ is an isomorphism. By Lemma 3.35 we can write $T_+ = W \oplus \mathbb{N}^q$, where W is a free $\mathbb{Z}$-module such that $\underline{\mathcal{E}}_{|W} = 0$ and, if we denote by $v_1, \dots, v_q$ the canonical base of $\mathbb{N}^q$, $\mathcal{E}^j(v_i) = \delta_{i,j}$. Consider the diagram:

$$\begin{tikzcd}
\mathbb{N}^q \oplus T \arrow[d, equal] & T_+ \arrow[d, equal] \\
\mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \arrow[r, "\gamma"] \arrow[d, "\sigma_{\underline{\mathcal{E}}}"'] & W \oplus \mathbb{N}^q \arrow[d, "\phi"] \\
\mathbb{Z}^q \oplus \mathbb{Z}^r \arrow[r, "\delta"] & \mathbb{Z}^r
\end{tikzcd}$$

$$\gamma(e_i) = v_i, \; \gamma_{|W} = -\mathrm{id}_W, \; \gamma(v_i) = 0$$

$$\delta(e_i) = \phi(v_i), \; \delta_{|\mathbb{Z}^r} = \mathrm{id}_{\mathbb{Z}^r}$$

One can check directly its commutativity. In this way, we get a map $s : \mathcal{X}_\phi \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}$. Again a direct computation on the diagrams defining s and $\pi_{\underline{\mathcal{E}}}$ shows that $\pi_{\underline{\mathcal{E}}} \circ s \simeq \mathrm{id}_{\mathcal{X}_\phi}$ and that the diagram inducing $G = s \circ \pi_{\underline{\mathcal{E}}}$ is

$$\begin{tikzcd}
\mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \arrow[r, "\alpha"] \arrow[d, "\sigma_{\underline{\mathcal{E}}}"'] & \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \arrow[d, "\sigma_{\underline{\mathcal{E}}}"] \\
\mathbb{Z}^q \oplus \mathbb{Z}^r \arrow[r, "\beta"] & \mathbb{Z}^q \oplus \mathbb{Z}^r
\end{tikzcd}$$

$$\alpha(e_i) = e_i - v_i, \; \alpha_{|W} = \mathrm{id}_W, \; \alpha_{|\mathbb{Z}^q} = 0$$

$$\beta(e_i) = \phi(v_i), \; \beta_{|\mathbb{Z}^r} = \mathrm{id}_{\mathbb{Z}^r}$$

We will prove that $G \simeq \mathrm{id}_{\mathcal{F}_{\underline{\mathcal{E}}}}$. An object of $\mathcal{F}_{\underline{\mathcal{E}}}(A)$, where A is a ring, coming from the atlas is given by $a = (\underline{z}, \lambda, \underline{\mu}) : \mathbb{N}^q \oplus W \oplus \mathbb{Z}^q \longrightarrow A$, where $\underline{z} = (a(e_i))_i = z_1, \dots, z_q \in A$, $\lambda = a_{|W} : W \longrightarrow A^*$ is an homomorphism and $\underline{\mu} = (\mu(v_i))_i = \mu_1 \dots, \mu_q \in A^*$. Moreover, $Ga = a \circ \alpha$ is $((z_i/\mu_i)_i, \lambda, \underline{1})$. It is now easy to check that $(\underline{\mu}, 1) : Ga \longrightarrow a$ is an isomorphism and that this map defines an isomorphism $G \longrightarrow \mathrm{id}_{\mathcal{F}_{\underline{\mathcal{E}}}}$. ■

Corollary 3.40. If $\underline{\mathcal{E}}$ is a smooth sequence then $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathcal{Z}_\phi$ is an open immersion with image $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$. □

It turns out that if $\underline{\mathcal{E}}$ is a smooth sequence, then $\mathcal{X}_\phi^{\underline{\mathcal{E}}}$ has a more explicit description:

Proposition 3.41. Let $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r$ be a smooth sequence, k be a field and $a \in \mathcal{X}_\phi(k)$. Then

$$a \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}(k) \iff \exists \mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}, \lambda : T \longrightarrow \bar{k}^* \text{ s.t. } a = \lambda 0^{\mathcal{E}}.$$

Moreover if $\lambda 0^{\mathcal{E}} \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}(k)$, for some $\mathcal{E} \in T_+^\vee$, $\lambda : T \longrightarrow \bar{k}^*$, then $\mathcal{E} \in \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}$. □

Proof. We can assume k algebraically closed and T_+ integral. In this case, $a \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}(k)$ if and only if $a : T_+ \longrightarrow k$ extends to a map $\operatorname{Ker} \underline{\mathcal{E}} \oplus \mathbb{N}^r = T_+^{\underline{\mathcal{E}}} \longrightarrow k$. So $\Leftarrow$ holds. Conversely, from Theorem 3.21, we can write $a = \lambda 0^{\mathcal{E}}$ where $\lambda : T \longrightarrow k^*$ and $\mathcal{E} \in (T_+^{\underline{\mathcal{E}}})^\vee$. From Lemma 3.35, we see that $T_+^{\underline{\mathcal{E}}\vee} = \langle \mathcal{E}^1, \dots, \mathcal{E}^r \rangle_{\mathbb{N}}$. Finally, if $\lambda 0^{\mathcal{E}} \in \mathcal{X}_\phi^{\underline{\mathcal{E}}}$ for some $\mathcal{E}$, then $\operatorname{Supp} \mathcal{E} \subseteq \operatorname{Supp} \underline{\mathcal{E}}$ and we are done. ■

Lemma 3.42. Let $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$ be a sequence of distinct smooth extremal rays and Θ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Set

$$\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta} = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}} \;\middle|\; \begin{array}{c} V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \neq \emptyset \\ \text{iff } \exists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\}.$$

Then, taking into account the identification made in Remark 3.16, we have

$$\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta} = \bigcup_{\underline{\delta} \in \Theta} \mathcal{F}_{\underline{\delta}}.$$

□

Proof. Let $\chi = (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \bigcup_{\underline{\delta} \in \Theta} \mathcal{F}_{\underline{\delta}}(T)$, for some scheme T and let $p \in V(z_{i_1}) \cap \cdots \cap V(z_{i_s})$. This means that the pullback of $\pi_{\underline{\mathcal{E}}}(\chi)$ to $\overline{k(p)}$ is given by $a = b0^{\mathcal{E}^{i_1} + \cdots + \mathcal{E}^{i_r}}$ for some $b : T_+ \longrightarrow \overline{k(p)}$. By definition, there exists $\underline{\delta} \in \Theta$ such that $a \in \mathcal{F}_{\underline{\delta}}(\overline{k(p)})$, that is, $a = \mu 0^{\delta}$ for some $\delta \in \langle \underline{\delta} \rangle_{\mathbb{N}}$, $\mu : T \longrightarrow \overline{k(p)}^*$. So

$$\operatorname{Supp} \mathcal{E}^{i_j} \subseteq \{a = 0\} = \operatorname{Supp} \delta \subseteq \operatorname{Supp} \underline{\delta} \Longrightarrow \mathcal{E}^{i_j} \in \langle \underline{\delta} \rangle_{\mathbb{N}}.$$

For the other inclusion, since all the $\mathcal{F}_{\underline{\delta}}$ are open substacks of $\mathcal{F}_{\underline{\mathcal{E}}}$, we can reduce the problem to the case of an algebraically closed field k. So let $(\underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}(k)$ and set $J = \{i \in I \mid z_i = 0\}$. By definition of $\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}$ there exists $\underline{\delta} \in \Theta$ such that $\underline{\eta} = (\mathcal{E}^j)_{j \in J} \subseteq \underline{\delta}$ and, taking into account Remark 3.16, this means that $a \in \mathcal{F}_{\underline{\eta}}(k) \subseteq \mathcal{F}_{\underline{\delta}}(k)$. ■

Definition 3.43. Let Θ be a collection of smooth sequences. We define

$$X_\phi^\Theta = \bigcup_{\underline{\delta}\in\Theta} \operatorname{Spec}\mathbb{Z}[T_+^{\underline{\delta}}] \subseteq \operatorname{Spec}\mathbb{Z}[T_+] \quad \text{and} \quad \mathcal{X}_\phi^\Theta = \bigcup_{\underline{\delta}\in\Theta} \mathcal{X}_\phi^{\underline{\delta}} \subseteq \mathcal{Z}_\phi. \qquad \square$$

Theorem 3.44. Let $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i\in I}$ be a sequence of distinct smooth extremal rays and Θ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Then we have an isomorphism

$$\mathcal{F}_{\underline{\mathcal{E}}}^\Theta = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{X}_\phi^\Theta) \xrightarrow{\simeq} \mathcal{X}_\phi^\Theta. \qquad \square$$

Proof. Taking into account Lemma 3.42, it is enough to note that

$$\pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{X}_\phi^\Theta) = \pi_{\underline{\mathcal{E}}}^{-1}\left(\bigcup_{\underline{\delta}\in\Theta} \mathcal{X}_\phi^{\underline{\delta}}\right) = \bigcup_{\underline{\delta}\in\Theta} \mathcal{F}_{\underline{\mathcal{E}}\cap\underline{\delta}} = \bigcup_{\underline{\delta}\in\Theta} \mathcal{F}_{\underline{\delta}} \xrightarrow{\simeq} \mathcal{X}_\phi^\Theta. \qquad \blacksquare$$

Proposition 3.45. Let $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i\in I}$ be a sequence of distinct smooth extremal rays and Θ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Then the set

$$\Delta^\Theta = \{\langle \eta_1, \dots, \eta_r\rangle_{\mathbb{Q}_+} \mid \exists \underline{\delta} \in \Theta \text{ s.t. } \eta_1, \dots, \eta_r \subseteq \underline{\delta}\}$$

is a toric fan in $T^\vee \otimes \mathbb{Q}$ whose associated toric variety over $\mathbb{Z}$ is X_ϕ^Θ. Moreover,

$$\mathcal{X}_\phi^\Theta \simeq [X_\phi^\Theta / \mathrm{D}(\mathbb{Z}^r)]. \qquad \square$$

Proof. We know that if $\underline{\eta}$ is a smooth sequence then $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\eta}}]$ is a smooth open subset of $\operatorname{Spec}\mathbb{Z}[T_+^{\text{int}}]$ and it is the affine toric variety associated to the cone $\langle\underline{\eta}\rangle_{\mathbb{Q}_+}$. It is then easy to check that Δ^Θ is a fan whose associated toric variety is X_ϕ^Θ. Since $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\eta}}]$ is the equivariant open subset of $\operatorname{Spec}\mathbb{Z}[T_+^{\text{int}}]$ inducing $\mathcal{X}_\phi^{\underline{\eta}}$ in $\mathcal{Z}_\phi$, then X^Θ is the equivariant open subset of $\operatorname{Spec}\mathbb{Z}[T_+^{\text{int}}]$ inducing $\mathcal{X}_\phi^\Theta$. In particular, we obtain the last isomorphism. $\blacksquare$

Lemma 3.46. Assume that T_+ is integral and set Θ for the set of all smooth sequences. Then X_ϕ^Θ is the smooth locus of $\operatorname{Spec}\mathbb{Z}[T_+]$. In particular, $\mathcal{Z}_\phi^{\text{sm}} = \mathcal{X}_\phi^\Theta \simeq [X_\phi^\Theta/\mathrm{D}(\mathbb{Z}^r)]$. $\square$

Proof. From Lemma 3.35, we know that $\operatorname{Spec}\mathbb{Z}[T_+^{\underline{\mathcal{E}}}]$ is smooth over $\mathbb{Z}$ and it is an open subset of $\operatorname{Spec}\mathbb{Z}[T_+]$. So we focus on the converse. Since $\operatorname{Spec}\mathbb{Z}[T_+]$ is flat over $\mathbb{Z}$, we can replace $\mathbb{Z}$ by an algebraically closed field k. Let $p \in \operatorname{Spec} k[T_+]$ be a smooth point. In particular, p^{om} is smooth too. If $p^{\mathrm{om}}=0$ then $p\in\operatorname{Spec} k[T]$ and we have done. So we can assume $p^{\mathrm{om}}=p_{\mathcal{E}}$ for some $0\neq\mathcal{E}\in T_+^\vee$ thanks to Lemma 3.23. We claim that there exists a smooth sequence $\mathcal{E}^1,\dots,\mathcal{E}^q$ such that $\mathcal{E}\in\langle\mathcal{E}^1,\dots,\mathcal{E}^q\rangle_{\mathbb{N}}$. This is enough to conclude that $p\in\operatorname{Spec} k[T_+^{\underline{\mathcal{E}}}]$. Indeed if $x_w\in p$ for some $w\in\operatorname{Ker}\underline{\mathcal{E}}\cap T_+$ then it belongs to $p^{\mathrm{om}}=p_{\mathcal{E}}$ and so $\mathcal{E}(w)>0$, which is not our case.

So assume that we have $\mathcal{E}\in T_+^\vee$ such that $p_{\mathcal{E}}$ is a regular point. Set $W=\langle\operatorname{Ker}\mathcal{E}\cap T_+\rangle_{\mathbb{Z}}$ and $T'_+=T_++W$. Note that $\operatorname{Spec} k[T'_+]$ is an open subset of $\operatorname{Spec} k[T_+]$ that contains $p_{\mathcal{E}}$. Moreover, $k[T'_+]/p_{\mathcal{E}}=k[W]$. Let $v_1,\dots,v_q\in T_+$ be elements such that

$$T'_+=\langle v_1,\dots,v_q\rangle_{\mathbb{N}}+W \quad\text{and}\quad \mathcal{E}(v_i)>0$$

with q minimal. We claim that $M=p_{\mathcal{E}}/p_{\mathcal{E}}^2\simeq k[W]^q$, where $p_{\mathcal{E}}$ is thought in $k[T'_+]$. Indeed M is a k-vector space over the x_v, $v\in T'_+$ that satisfies: $\mathcal{E}(v)>0$ and whenever we have $v=v'+v''$ with $v',v''\in T'_+$ it follows that $\mathcal{E}(v')=0$ or $\mathcal{E}(v'')=0$. A simple computation shows that such a v must be of the form v_i+W for some i. But since we have chosen q minimal we have $(v_i+W)\cap(v_j+W)=\emptyset$ if $i\neq j$. This implies that M is a free $k[W]$-module with basis $x_{v_1},\dots,x_{v_q}$. This shows that $q=\operatorname{ht} p_{\mathcal{E}}$.

Now set $V=\langle v_1,\dots,v_q\rangle_{\mathbb{Z}}$. Since $V+W=T$, $\operatorname{rk} V\leq q$ and

$$k[W]\simeq k[T'_+]/p_{\mathcal{E}}\Longrightarrow \operatorname{rk} T=\dim k[T'_+]=\operatorname{ht} p_{\mathcal{E}}+\dim k[W]=q+\operatorname{rk} W,$$

we obtain that $v_1,\dots,v_q$ are independent. Let $\mathcal{E}^1,\dots,\mathcal{E}^q$ given by $\mathcal{E}^i(v_j)=\delta_{i,j}$ and $\mathcal{E}^i_{|W}=0$. In particular, $W=\operatorname{Ker}\underline{\mathcal{E}}$ and it is generated by elements in T_+. Since $\mathcal{E}_{|W}=0$, we have

$$\mathcal{E}=\sum_{i=1}^q\mathcal{E}(v_i)\mathcal{E}^i,\qquad \mathcal{E}(v_i)>0.$$

Moreover, since $T_+\subseteq T'_+$ and $\mathcal{E}^i\in T'^\vee_+$ we get that $\mathcal{E}^i\in T_+^\vee$, as required. ■

Theorem 3.47. If $\underline{\mathcal{E}}$ is a sequence of distinct indecomposable rays containing the smooth extremal rays then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence

$$\left\{(\underline{\mathcal{L}},\underline{\mathcal{M}},\underline{z},\delta)\in\mathcal{F}_{\underline{\mathcal{E}}}\;\middle|\;\begin{array}{c}V(z_{i_1})\cap\cdots\cap V(z_{i_s})=\emptyset\\ \text{if } \mathcal{E}^{i_1},\dots,\mathcal{E}^{i_s} \text{ is not a}\\ \text{smooth sequence}\end{array}\right\}=\pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{Z}_\phi^{\mathrm{sm}})\xrightarrow{\simeq}\mathcal{Z}_\phi^{\mathrm{sm}}.$$

□

Proof. Lemma 3.46 tells us that $\mathcal{Z}_\phi^{\mathrm{sm}} = \mathcal{X}_\phi^{\Theta}$, where Θ is the collection of all smooth sequences, while Lemma 3.39 allows us to replace $\underline{\mathcal{E}}$ with the sequence of all smooth extremal rays. Therefore, it is enough to apply Theorem 3.44 and Proposition 3.45. ■

Proposition 3.48. Let $a : T_+ \longrightarrow k \in \mathcal{X}_\phi(k)$, where k is a field. Then a lies in $\mathcal{Z}_\phi^{\mathrm{sm}}$ if and only if there exists a smooth ray $\mathcal{E} \in T_+^\vee$ and $\lambda : T \longrightarrow \bar{k}^*$ such that $a = \lambda 0^{\mathcal{E}}$. □

Proof. Apply Theorem 3.47 and Proposition 3.41. ■

3.4 Extension of objects from codimension 1

In this subsection, we want to explain how it is possible, in certain cases, to check that an object of $\mathcal{X}_\phi$ over a sufficiently regular scheme X comes (uniquely) from $\mathcal{F}_{\underline{\mathcal{E}}}$ only checking what happens in codimension 1.

Notation 3.49. Given a scheme X, we will denote by $\underline{\mathrm{Pic}}\, X$ the category whose objects are invertible sheaves and whose arrows are maps between them. □

Proposition 3.50. Let $X \xrightarrow{f} Y$ be a map of schemes. If $\underline{\mathrm{Pic}}\, Y \xrightarrow{f^*} \underline{\mathrm{Pic}}\, X$ is fully faithful (resp. an equivalence) then $\mathcal{X}_\phi(Y) \xrightarrow{f^*} \mathcal{X}_\phi(X)$ has the same property. □

Proof. Let $(\underline{\mathcal{L}}, a), (\underline{\mathcal{L}}', a') \in \mathcal{X}_\phi(Y)$ and $\underline{\sigma} : f^*(\underline{\mathcal{L}}, a) \longrightarrow f^*(\underline{\mathcal{L}}', a')$ be a map in $\mathcal{X}_\phi(X)$. Any map $\sigma_i : f^*\mathcal{L}_i \longrightarrow f^*\mathcal{L}_i$ comes from a unique map $\tau_i : \mathcal{L}_i \longrightarrow \mathcal{L}_i$, that is, $\sigma_i = f^*\tau_i$. Since

$$f^*(\underline{\tau}^{\phi(t)}(a(t))) = \underline{\sigma}^{\phi(t)}(f^*a(t)) = f^*(a'(t)) \Longrightarrow \underline{\tau}^{\phi(t)}(a(t)) = a'(t)$$

$\underline{\tau}$ is a map $(\underline{\mathcal{L}}, a) \longrightarrow (\underline{\mathcal{L}}', a')$ such that $f^*\underline{\tau} = \underline{\sigma}$. We can conclude that $f^* : \mathcal{X}_\phi(Y) \longrightarrow \mathcal{X}_\phi(X)$ is fully faithful.

Now assume that $\underline{\mathrm{Pic}}\, Y \xrightarrow{f^*} \underline{\mathrm{Pic}}\, X$ is an equivalence. We have to prove that $\mathcal{X}_\phi(Y) \xrightarrow{f^*} \mathcal{X}_\phi(X)$ is essentially surjective. So let $(\underline{\mathcal{M}}, b) \in \mathcal{X}_\phi(X)$. Since f^* is an equivalence, we can assume $\mathcal{M}_i = f^*\mathcal{L}_i$ for some invertible sheaf $\mathcal{L}_i$ on Y. Since for any invertible sheaf $\mathcal{L}$ on Y one has that $\mathcal{L}(Y) \simeq (f^*\mathcal{L})(X)$, any section $b(t) \in \underline{\mathcal{M}}^{\phi(t)}$ extends to a unique section $a(t) \in \underline{\mathcal{L}}^{\phi(t)}$. Since

$$f^*(a(t) \otimes a(s)) = b(t) \otimes b(s) = b(t+s) = f^*(a(t+s)) \Longrightarrow a(t) \otimes a(s) = a(t+s)$$

for any $t, s \in T_+$ and $a(0) = 1$, it follows that $(\underline{\mathcal{L}}, a) \in \mathcal{X}_\phi(Y)$ and $f^*(\underline{\mathcal{L}}, a) = (\underline{\mathcal{M}}, b)$. ■

Corollary 3.51. Let $X \xrightarrow{f} Y$ be a map of schemes and consider a commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\
Y \arrow[r] \arrow[ru, dashed] & \mathcal{X}_\phi
\end{tikzcd}$$

where $\underline{\mathcal{E}}$ is a sequence of elements of $T_+^\vee$. Then if $\underline{\mathrm{Pic}}\, X \xrightarrow{f^*} \underline{\mathrm{Pic}}\, Y$ is fully faithful (resp. an equivalence) the dashed lifting is unique (resp. exists). □

Proof. It is enough to consider the 2-commutative diagram:

$$\begin{tikzcd}
\mathcal{F}_{\underline{\mathcal{E}}}(Y) \arrow[r, "f^*"] \arrow[d, "\pi_{\underline{\mathcal{E}}}"'] & \mathcal{F}_{\underline{\mathcal{E}}}(X) \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\
\mathcal{X}_\phi(Y) \arrow[r, "f^*"] & \mathcal{X}_\phi(X)
\end{tikzcd}$$

and note that f^* is fully faithful (resp. an equivalence) in both cases. ■

Theorem 3.52. Let X be a locally noetherian and locally factorial scheme, $\underline{\mathcal{E}} = (\mathcal{E}^i)_{i \in I}$ be a sequence of distinct smooth extremal rays and Θ be a collection of smooth sequences with rays in $\underline{\mathcal{E}}$. Consider the full subcategories

$$\mathcal{C}_X^\Theta = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \;\middle|\; \begin{array}{c} \operatorname{codim}_X V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \geq 2 \\ \text{if } \nexists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1}, \ldots, \mathcal{E}^{i_s} \subseteq \underline{\delta} \end{array} \right\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

and

$$\mathcal{D}_X^\Theta = \left\{ \chi \in \mathcal{X}_\phi(X) \;\middle|\; \begin{array}{c} \forall p \in X \text{ with } \operatorname{codim}_p X \leq 1 \\ \chi_{|\overline{k(p)}} \in \mathcal{X}_\phi^\Theta \end{array} \right\} \subseteq \mathcal{X}_\phi(X).$$

Then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories

$$\mathcal{C}_X^\Theta = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{D}_X^\Theta) \xrightarrow{\simeq} \mathcal{D}_X^\Theta.$$ □

Proof. We claim that

$$\mathcal{C}_X^\Theta = \{\chi \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \exists U \subseteq X \text{ open subset s.t. } \operatorname{codim}_X X - U \geq 2,\ \chi_{|U} \in \mathcal{F}_{\underline{\mathcal{E}}}^\Theta(U)\}.$$

$\subseteq$ Taking into account the definition of $\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}$ in Lemma 3.42, it is enough to consider

$$U = X - \bigcup_{\nexists \underline{\delta} \in \Theta \text{ s.t. } \mathcal{E}^{i_1},\dots,\mathcal{E}^{i_s} \subseteq \underline{\delta}} V(z_{i_1}) \cap \cdots \cap V(z_{i_s}).$$

$\supseteq$ If $p \in V(z_{i_1}) \cap \cdots \cap V(z_{i_s})$ and $\operatorname{codim}_p X \leq 1$ then $p \in U$ and again by definition of $\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}$ there exists $\underline{\delta} \in \Theta$ such that $\mathcal{E}^{i_1}, \dots, \mathcal{E}^{i_s} \subseteq \underline{\delta}$.

We also claim that

$$\mathcal{D}_X^{\Theta} = \{\chi \in \mathcal{X}_\phi(X) \mid \exists U \subseteq X \text{ open subset s.t. } \operatorname{codim}_X X - U \geq 2,\ \chi_{|U} \in \mathcal{X}_\phi^{\Theta}(U)\}.$$

$\supseteq$ Such a U contains all the codimension 1 or 0 points of X.

$\subseteq$ Let $\chi \in \mathcal{D}_X^{\Theta}$ and $X \xrightarrow{g} \mathcal{X}_\phi$ be the induced map. If ξ is a generic point of X, we know that $f(\xi) \in |\mathcal{X}_\phi^{\Theta}| \subseteq |\mathcal{Z}_\phi|$. In particular, $f(|X|) \subseteq |\mathcal{Z}_\phi|$. Since both X and $\mathcal{Z}_\phi$ are reduced g factors through a map $X \xrightarrow{g} \mathcal{Z}_\phi$. Since $\mathcal{X}_\phi^{\Theta}$ is an open substack of $\mathcal{Z}_\phi$, it follows that $U = g^{-1}(\mathcal{X}_\phi^{\Theta})$ is an open subscheme of X, $\chi_{|U} \in \mathcal{X}_\phi^{\Theta}(U)$ and, by definition of $\mathcal{D}_X^{\Theta}$, $\operatorname{codim}_X X - U \geq 2$.

Taking into account Theorem 3.44, it is clear that $\mathcal{C}_X^{\Theta} = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{D}_X^{\Theta})$. We will make use of the fact that if $U \subseteq X$ is an open subscheme such that $\operatorname{codim}_X X - U \geq 2$ then the restriction yields an equivalence $\underline{\operatorname{Pic}}\, X \simeq \underline{\operatorname{Pic}}\, U$. The map $\mathcal{C}_X^{\Theta} \longrightarrow \mathcal{D}_X^{\Theta}$ is essentially surjective since, given an object of $\mathcal{D}_X^{\Theta}$, the associated map $X \xrightarrow{g} \mathcal{X}_\phi$ fits in a 2-commutative diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \mathcal{F}_{\underline{\mathcal{E}}}^{\Theta} \subseteq \mathcal{F}_{\underline{\mathcal{E}}} \arrow[d, "\pi_{\underline{\mathcal{E}}}"] \\
X \arrow[r, "g"] & \mathcal{X}_\phi
\end{tikzcd}$$

and so lifts to a map $X \longrightarrow \mathcal{F}_{\underline{\mathcal{E}}}$ thanks to Corollary 3.51.

It remains to show that $\mathcal{C}_X^{\Theta} \longrightarrow \mathcal{D}_X^{\Theta}$ is fully faithful. Let $\chi, \chi' \in \mathcal{C}_X^{\Theta}$ and U, U' be the open subscheme given in the definition of $\mathcal{C}_X^{\Theta}$. Set $V = U \cap U'$. Taking into account Proposition 3.50 and Theorem 3.44, we have

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{F}_{\underline{\mathcal{E}}}(X)}(\chi, \chi') \arrow[r] \arrow[d, phantom, "\wr"] & \operatorname{Hom}_{\mathcal{X}_\phi(X)}(\chi, \chi') \arrow[d, phantom, "\wr"] \\
\operatorname{Hom}_{\mathcal{F}_{\underline{\mathcal{E}}}(V)}(\chi_{|V}, \chi'_{|V}) \arrow[r] \arrow[d, phantom, "\wr"] & \operatorname{Hom}_{\mathcal{X}_\phi(V)}(\chi_{|V}, \chi'_{|V}) \arrow[d, phantom, "\wr"] \\
\operatorname{Hom}_{\mathcal{F}_{\underline{\mathcal{E}}}^{\Theta}(V)}(\chi_{|V}, \chi'_{|V}) \arrow[r, "\simeq"'] & \operatorname{Hom}_{\mathcal{X}_\phi^{\Theta}(V)}(\chi_{|V}, \chi'_{|V}).
\end{tikzcd}$$

■

4 Galois Covers for a Diagonalizable Group

In this section, we will fix a finite diagonalizable group scheme G over $\mathbb{Z}$ and we will call $M = \operatorname{Hom}(G, \mathbb{G}_m)$ its character group. So M is a finite abelian group and $G = \mathrm{D}(M)$. With abuse of notation, we will write $\mathcal{O}_U[M] = \mathcal{O}_U[G_U]$ and $\mathcal{Z}_M = \mathcal{Z}_{\mathrm{D}(M)}$, the main component of $\mathrm{D}(M)$-Cov. It turns out that in this case $\mathrm{D}(M)$-covers have a nice and more explicit description.

In the first subsection, we will show that $\mathrm{D}(M)\text{-Cov} \simeq \mathcal{X}_\phi$ for an explicit map $T_+ \xrightarrow{\phi} \mathbb{Z}^M/\langle e_0\rangle$ and that this isomorphism preserves the main irreducible components of both stacks. Moreover, we will study the connection between $\mathrm{D}(M)$-Cov and the equivariant Hilbert schemes $M\text{-Hilb}^{\underline{m}}$ and prove some results about their geometry.

Then we will introduce an upper semicontinuous map $|\mathrm{D}(M)\text{-Cov}| \xrightarrow{h} \mathbb{N}$ that yields a stratification by open substacks of $\mathrm{D}(M)$-Cov. We will also see that $\{h = 0\}$ coincides with the open substack of $\mathrm{D}(M)$-torsors, while $\{h \leq 1\}$ lies in the smooth locus of $\mathcal{Z}_M$ and can be described by a particular set of smooth extremal rays. This will allow us to describe the $\mathrm{D}(M)$-covers over a locally noetherian and locally factorial scheme X with $(\operatorname{char} X, |M|) = 1$ whose total space is regular in codimension 1 (which, a posteriori, is equivalent to the normal condition).

4.1 The stack D(M)-Cov and its main irreducible component $\mathcal{Z}_M$

Consider a scheme U and a cover $X = \operatorname{Spec} \mathcal{A}$ on it. An action of $\mathrm{D}(M)$ on it consists of a decomposition

$$\mathcal{A} = \bigoplus_{m \in M} \mathcal{A}_m$$

such that $\mathcal{O}_U \subseteq \mathcal{A}_0$ and the multiplication maps $\mathcal{A}_m \otimes \mathcal{A}_n$ into $\mathcal{A}_{m+n}$. If X/U is a $\mathrm{D}(M)$-cover there exists an fppf covering $\{U_i \longrightarrow U\}$ such that $\mathcal{A}_{|U_i} \simeq \mathcal{O}_{U_i}[M]$ as $\mathrm{D}(M)$-comodules. This means that for any $m \in M$ we have

$$\forall\, i\ (\mathcal{A}_m)_{|U_i} \simeq \mathcal{O}_{U_i} \Longrightarrow \mathcal{A}_m \text{ invertible.}$$

Conversely, any M-graded quasi-coherent algebra $\mathcal{A} = \bigoplus_{m\in M} \mathcal{A}_m$ with $\mathcal{A}_0 = \mathcal{O}_U$ and $\mathcal{A}_m$ invertible for any m yields a $\mathrm{D}(M)$-cover $\operatorname{Spec} \mathcal{A}$.

So the stack $\mathrm{D}(M)$-Cov can be described as follows. An object of $\mathrm{D}(M)\text{-Cov}(U)$ is given by a collection of invertible sheaves $\mathcal{L}_m$ for $m \in M$ with maps

$$\psi_{m,n} : \mathcal{L}_m \otimes \mathcal{L}_n \longrightarrow \mathcal{L}_{m+n}$$

and an isomorphism $\mathcal{O}_U \simeq \mathcal{L}_0$ satisfying the following relations:

Commutativity

$$\begin{tikzcd}
\mathcal{L}_m \otimes \mathcal{L}_n \arrow[rr, "\simeq"] \arrow[dr, "\psi_{m,n}"'] & & \mathcal{L}_n \otimes \mathcal{L}_m \arrow[dl, "\psi_{n,m}"] \\
& \mathcal{L}_{m+n} &
\end{tikzcd}$$

Associativity

$$\begin{tikzcd}
\mathcal{L}_m \otimes \mathcal{L}_n \otimes \mathcal{L}_t \arrow[r, "\mathrm{id}\otimes\psi_{n,t}"] \arrow[d, "\psi_{m,n}\otimes\mathrm{id}"'] & \mathcal{L}_m \otimes \mathcal{L}_{n+t} \arrow[d, "\psi_{m,n+t}"] \\
\mathcal{L}_{m+n} \otimes \mathcal{L}_t \arrow[r, "\psi_{m+n,t}"] & \mathcal{L}_{m+n+t}
\end{tikzcd}$$

Neutral Element

$$\begin{tikzcd}
\mathcal{L}_m \arrow[r, "\simeq"] \arrow[rrr, bend right, "\mathrm{id}"'] & \mathcal{L}_m \otimes \mathcal{O}_U \arrow[r, "\simeq"] & \mathcal{L}_m \otimes \mathcal{L}_0 \arrow[r, "\psi_{m,0}"] & \mathcal{L}_m
\end{tikzcd}$$

If we assume that $\mathcal{L}_m = \mathcal{O}_U v_m$, that is, that we have sections v_m generating $\mathcal{L}_m$, the maps $\psi_{m,n}$ can be thought of as elements of $\mathcal{O}_U$ and the algebra structure is given by $v_m v_n = \psi_{m,n} v_{m+n}$. In this case, we can rewrite the above conditions obtaining

$$\psi_{m,n} = \psi_{n,m}, \quad \psi_{m,0} = 1, \quad \psi_{m,n}\psi_{m+n,t} = \psi_{n,t}\psi_{n+t,m}. \tag{4.1}$$

The functor that associates to a scheme U the functions $\psi : M \times M \longrightarrow \mathcal{O}_U$ satisfying the above conditions is clearly representable by the spectrum of the ring

$$R_M = \mathbb{Z}[x_{m,n}]/(x_{m,n} - x_{n,m}, x_{m,0} - 1, x_{m,n}x_{m+n,t} - x_{n,t}x_{n+t,m}). \tag{4.2}$$

In this way, we obtain a Zariski epimorphism Spec $R_M \longrightarrow \mathrm{D}(M)$-Cov that we will prove to be smooth. We now want to prove that the stack $\mathrm{D}(M)$-Cov is isomorphic to a stack of the form $\mathcal{X}_\phi$.

Definition 4.1. Define $\tilde{K}_+$ as the quotient monoid of $\mathbb{N}^{M\times M}$ by the equivalence relation generated by

$$e_{m,n} \sim e_{n,m}, \quad e_{m,0} \sim 0, \quad e_{m,n} + e_{m+n,t} \sim e_{n,t} + e_{n+t,m}.$$

Also define $\phi_M : \tilde{K}_+ \longrightarrow \mathbb{Z}^M/\langle e_0\rangle$ by $\phi_M(e_{m,n}) = e_m + e_n - e_{m+n}$. □

Proposition 4.2. $R_M \simeq \mathbb{Z}[\tilde{K}_+]$ and there exists an isomorphism

$$\mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)\text{-Cov} \tag{4.3}$$

such that $\operatorname{Spec}\mathbb{Z}[\tilde{K}_+] \simeq \operatorname{Spec} R_M \longrightarrow \mathrm{D}(M)\text{-Cov} \simeq \mathcal{X}_{\phi_M}$ is the map defined in Proposition 3.3. In particular,

$$\mathrm{D}(M)\text{-Cov} \simeq [\operatorname{Spec} R_M/\mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)]. \qquad \square$$

Proof. The required isomorphism sends $(\underline{\mathcal{L}}, \tilde{K}_+ \xrightarrow{\psi} \mathrm{Sym}^*\underline{\mathcal{L}}) \in \mathcal{X}_{\phi_M}$ to the object of $\mathrm{D}(M)$-Cov given by invertible sheaves $(\mathcal{L}'_m = \mathcal{L}_m^{-1})$ and $\psi_{m,n} = \psi(e_{m,n})$. ■

We want to prove that the isomorphism (4.3) sends $\mathcal{Z}_{\phi_M}$ to $\mathcal{Z}_M$ (see Definition 2.3) and $\mathcal{B}_{\phi_M}$ to $\mathrm{BD}(M)$. We need the following classical result on the structure of a $\mathrm{D}(M)$-torsor (see [8, Exposé VIII, Proposition 4.1 and 4.6]):

Proposition 4.3. Let M be a finite abelian group and $P \longrightarrow U$ a $\mathrm{D}(M)$-equivariant map. Then P is an fppf $\mathrm{D}(M)$-torsor if and only if $P \in \mathrm{D}(M)\text{-Cov}(U)$ and all the multiplication maps $\psi_{m,n}$ are isomorphisms. □

Now consider the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & \mathbb{Z}^M/\langle e_0\rangle \arrow[r] & M \arrow[r] & 0 \\
 & & e_m \arrow[r, mapsto] & m &
\end{tikzcd}$$

Definition 4.4. For $m, n \in M$, we define

$$v_{m,n} = \phi_M(e_{m,n}) = e_m + e_n - e_{m+n} \in K$$

and K_+ as the submonoid of K generated by the $v_{m,n}$. We will set $x_{m,n} = x^{v_{m,n}} \in \mathbb{Z}[K_+]$ and, for $\mathcal{E} \in K_+^\vee$, $\mathcal{E}_{m,n} = \mathcal{E}(v_{m,n})$. □

Lemma 4.5. The map

$$\begin{tikzcd}
\tilde{K}_+ \arrow[r] & K \\
e_{m,n} \arrow[r, mapsto] & v_{m,n}
\end{tikzcd}$$

is the associated group of $\tilde{K}_+$ and K_+ is its associated integral monoid. In particular, we have a 2-cartesian diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{Z}[K] \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}[K_+] \arrow[r] \arrow[d] & \operatorname{Spec} R_M \arrow[d] \\
\mathrm{BD}(M) \arrow[r] & \mathcal{Z}_M \arrow[r] & \mathrm{D}(M)\text{-Cov}
\end{tikzcd}$$

□

Proof. Set $x=\prod_{m,n} x_{m,n}$. Since an object $\psi \in \operatorname{Spec} R_M(U)$ is a torsor if and only if $\psi_{m,n} \in \mathcal{O}_U^*$ for all m, n, it follows that $(\operatorname{Spec} R_M)_x = \mathrm{BD}(M) \times_{\mathrm{D}(M)\text{-Cov}} \operatorname{Spec} R_M$. We want to define an inverse to $(R_M)_x \longrightarrow \mathbb{Z}[K]$. Consider the algebra S_M over R_M induced by the atlas map $\operatorname{Spec} R_M \longrightarrow \mathrm{D}(M)\text{-Cov}$, that is,

$$S_M = \bigoplus_{m\in M} R_M w_m \quad \text{with } w_0 = 1, \quad w_m w_n = x_{m,n} w_{m+n}$$

The algebra $(S_M)_x$ is a $\mathrm{D}(M)$-torsor over $(R_M)_x$ and so $w_m \in (S_M)_x^*$ for all m. In particular, we can define a group homomorphism

$$\begin{tikzcd}
\mathbb{Z}^M/\langle e_0\rangle \arrow[r] & (S_M)_x^* \\
e_m \arrow[r, mapsto] & w_m
\end{tikzcd}$$

which restricts to a map $K \longrightarrow (R_M)_x$ that sends $v_{m,n}$ to $x_{m,n}$. In particular, the map $\tilde{K}_+ \longrightarrow K$ defined in the statement gives the associated group of $\tilde{K}_+$ and has as image exactly K_+, which means that K_+ is the integral monoid associated to $\tilde{K}_+$.

In order to conclude the proof it is enough to apply Remark 3.8 and Proposition 3.9. ■

Corollary 4.6. The isomorphism $\mathcal{X}_{\phi_M} \simeq \mathrm{D}(M)\text{-Cov}$ (4.3) induces isomorphisms $\mathcal{B}_{\phi_M} \simeq \mathrm{BD}(M)$ and $\mathcal{Z}_{\phi_M} \simeq \mathcal{Z}_M$. In particular, $\mathcal{Z}_M$ is an irreducible component of $\mathrm{D}(M)$-Cov and

$$\mathrm{BD}(M) \simeq [\operatorname{Spec}\mathbb{Z}[K]/\mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)] \quad \text{and} \quad \mathcal{Z}_M \simeq [\operatorname{Spec}\mathbb{Z}[K_+]/\mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)]$$

□

Note that the induced map $\phi_M : K \longrightarrow \mathbb{Z}^M/\langle e_0\rangle$ is just the inclusion and so it is injective. This means that any result obtained in Section 3 applies naturally in the context of D(M)-covers. In particular, now we show how we can describe the objects of $\mathcal{F}_{\underline{\mathcal{E}}}$, for a sequence of rays in $\tilde{K}_+^\vee$, in a simpler way.

Proposition 4.7. Let $M \simeq \prod_{i=1}^n \mathbb{Z}/l_i\mathbb{Z}$ be a decomposition and let $m_1, \dots, m_n$ be the associated generators. Given $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$ define $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$ as the stack whose objects over a scheme X are sequences $\underline{\mathcal{L}} = \mathcal{L}_1, \dots, \mathcal{L}_n$, $\underline{\mathcal{M}} = \mathcal{M}_1, \dots, \mathcal{M}_r$, $\underline{z} = z_1, \dots, z_r$, $\underline{\mu} = \mu_1, \dots, \mu_n$ where $\underline{\mathcal{L}}$ and $\underline{\mathcal{M}}$ are invertible sheaves over X, $z_i \in \mathcal{M}_i$ and $\underline{\mu}$ are isomorphisms

$$\mu_i : \mathcal{L}_i^{-l_i} \xrightarrow{\simeq} \underline{\mathcal{M}}^{\underline{\mathcal{E}}(l_i e_{m_i})} = \mathcal{M}_1^{\mathcal{E}^1(l_i e_{m_i})} \otimes \cdots \otimes \mathcal{M}_r^{\mathcal{E}^r(l_i e_{m_i})}.$$

Then we have an isomorphism of stacks

$$\begin{tikzcd}
\mathcal{F}_{\underline{\mathcal{E}}} \arrow[r] & \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}} \\
(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \arrow[r, mapsto] & ((\mathcal{L}_{m_i})_{i=1,\dots,n}, \underline{\mathcal{M}}, \underline{z}, (\lambda(l_i e_{m_i}))_{i=1,\dots,n})
\end{tikzcd}$$ □

Proof. We want to find $\sigma, V, v_1, \dots, v_q$ as in Definition 3.25 such that $\mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red},\sigma} = \mathcal{F}_{\underline{\mathcal{E}}}^{\mathrm{red}}$ and that the map in the statement coincides with the one defined in Proposition 3.27. Set $\delta^i : M \longrightarrow \{0, \dots, l_i - 1\}$ as the map such that $\pi_i(m) = \pi_i(\delta_m^i m_i)$, where $\pi_i : M \longrightarrow \mathbb{Z}/l_i\mathbb{Z}$ is the projection, and think of it also as a map $\delta^i : \mathbb{Z}^M/\langle e_0\rangle \longrightarrow \mathbb{Z}$. Set $V = \bigoplus_{i=1}^n \mathbb{Z}e_{m_i}$, $v_i = e_{m_i}$ and $\sigma : \mathbb{Z}^M/\langle e_0\rangle \longrightarrow V$ as $\sigma(e_m) = \sum_{i=1}^n \delta_m^i v_i$. Clearly, $(\mathrm{id} - \sigma)\mathbb{Z}^M/\langle e_0\rangle \subseteq K$ and $(\mathrm{id} - \sigma)V = 0$. So $W = \sigma K$. We have

$$\sigma(v_{m,n}) = \sum_{i=1}^n \delta_{m,n}^i v_i \in \bigoplus_{i=1}^n l_i\mathbb{Z}v_i$$

since $\delta_{m,n}^i \in \{0, l_i\}$ for all i. On the other hand, $\sigma(v_{(l_i-1)m_i, m_i}) = l_i v_i$. Therefore, we have $W = \bigoplus_{i=1}^n l_i\mathbb{Z}v_i$. It is now easy to check that all the definitions agree. ■

We now want to express the relation between D(M)-Cov and the equivariant Hilbert scheme, which can be defined as follows. Given $\underline{m} = m_1, \dots, m_r \in M$, so that D$(M)$ acts on $\mathbb{A}_{\mathbb{Z}}^r = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_r]$ with graduation $\deg x_i = m_i$, we define $M\text{-Hilb}^{\underline{m}} : \mathrm{Sch}^{\mathrm{op}} \longrightarrow$ (sets) as the functor that associates to a scheme Y the set of pairs $(X \xrightarrow{f} Y, j)$, where $X \in$ D(M)-Cov(Y) and $j : X \longrightarrow \mathbb{A}_Y^r$ is an equivariant closed immersion over Y. Such a pair can also be thought of as a coherent sheaf of algebras $\mathcal{A} \in$ D(M)-Cov(Y) together with a graded surjective map $\mathcal{O}_Y[x_1, \dots, x_r] \longrightarrow \mathcal{A}$. This functor is proved to be a scheme of finite type in [10].

Proposition 4.8. Let $\underline{m} = m_1, \dots, m_r \in M$. The forgetful map $\vartheta_{\underline{m}} : M\text{-Hilb}^{\underline{m}} \longrightarrow \mathrm{D}(M)\text{-Cov}$ is a smooth Zariski epimorphism onto the open substack $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$ of $\mathrm{D}(M)\text{-Cov}$ of sheaves of algebras $\mathscr{A}$ such that, for all $y \in Y$, $\mathscr{A} \otimes k(y)$ is generated in the degrees $m_1, \dots, m_r$ as a $k(y)$-algebra. Moreover, $M\text{-Hilb}^{\underline{m}}$ is an open subscheme of a vector bundle over $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$. □

Proof. Let $\mathscr{A} = \bigoplus_{m \in M} \mathscr{A}_m \in \mathrm{D}(M)\text{-Cov}$ and consider the map

$$\eta_{\mathscr{A}} : \mathrm{Sym}(\mathscr{A}_{m_1} \oplus \cdots \oplus \mathscr{A}_{m_r}) \longrightarrow \mathscr{A}$$

induced by the direct sum of the inclusions $\mathscr{A}_{m_i} \longrightarrow \mathscr{A}$. It is easy to check that $\eta_{\mathscr{A}}$ is surjective if and only if $\mathscr{A} \in \mathrm{D}(M)\text{-Cov}^{\underline{m}}$. Therefore, $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$ is an open substack of $\mathrm{D}(M)\text{-Cov}$ and clearly contains the image of $\vartheta_{\underline{m}}$. Consider now the cartesian diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & M\text{-Hilb}^{\underline{m}} \arrow[d, "\vartheta_{\underline{m}}"] \\ T \arrow[r, "\mathscr{A}"] & \mathrm{D}(M)\text{-Cov}^{\underline{m}} \end{tikzcd}$$

and let $U \xrightarrow{\phi} T$ be a map. The objects of $F(U)$ are pairs composed by a graded surjection $\mathcal{O}_U[x_1, \dots, x_r] \longrightarrow \mathscr{B}$ and an isomorphism $\mathscr{B} \simeq \phi^*\mathscr{A}$. This is equivalent to giving a graded surjection $\mathcal{O}_U[x_1, \dots, x_r] \longrightarrow \phi^*\mathscr{A}$. In this way, we obtain a map

$$F \xrightarrow{g_T} \prod_i \underline{\mathrm{Hom}}_T(\mathcal{O}_T, \mathscr{A}_{m_i}) \simeq \mathrm{Spec}\,\mathrm{Sym}\Big(\bigoplus_i \mathscr{A}_{m_i}^{-1}\Big).$$

We claim that this is an open immersion. Indeed given $(a_i)_i : U \longrightarrow \prod_i \underline{\mathrm{Hom}}_T(\mathcal{O}_T, \mathscr{A}_{m_i})$, the fiber product with F is the locus where the induced graded map $\mathcal{O}_U[x_1, \dots, x_r] \longrightarrow \mathscr{A} \otimes \mathcal{O}_U$ is surjective, that is an open subscheme of U. In particular, F is smooth over T and so $\vartheta_{\underline{m}}$ is smooth too. It is easy to check that it is also a Zariski epimorphism. Finally, the vector bundle $\mathscr{N}$ of the statement is defined over any $U \longrightarrow \mathrm{D}(M)\text{-Cov}^{\underline{m}}$ given by $\mathscr{A} = \bigoplus_m \mathscr{A}_m$ by $\mathscr{N}_{|U} = \bigoplus_i \mathscr{A}_{m_i}^{-1}$. ■

Remark 4.9. If the sequence $\underline{m}$ contains all elements of $M - \{0\}$, then $\mathrm{D}(M)\text{-Cov}^{\underline{m}} = \mathrm{D}(M)\text{-Cov}$. Therefore, in this case $M\text{-Hilb}^{\underline{m}}$ is an atlas for $\mathrm{D}(M)\text{-Cov}$. □

Remark 4.10. We have cartesian diagrams:

$$\begin{tikzcd}
W_{\underline{m}} \arrow[r] \arrow[d, "\mathrm{D}(\mathbb{Z}^M/\langle e_0\rangle)\ \text{torsors}"'] & V_{\underline{m}} \arrow[r] \arrow[d, "\text{open immersions}"'] & U_{\underline{m}} \arrow[r] \arrow[d, "\text{vector bundles}"'] & \operatorname{Spec} R_M \arrow[d, "\text{open immersions}"'] \\
M\text{-Hilb}^{\underline{m}} \arrow[r] & H_{\underline{m}} \arrow[r] & \mathrm{D}(M)\text{-Cov}^{\underline{m}} \arrow[r] & \mathrm{D}(M)\text{-Cov}
\end{tikzcd}$$

where the comments apply to the horizontal maps. In particular, since $\mathrm{BD}(M) \subseteq \mathrm{D}(M)\text{-Cov}^{\underline{m}}$, we can conclude that $\vartheta_{\underline{m}}^{-1}(\mathcal{Z}_M)$ is the main irreducible component of $M\text{-Hilb}^{\underline{m}}$. Moreover, the above diagram shows that $M\text{-Hilb}^{\underline{m}}$ and $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$, as well as their main irreducible components, share many properties such as smoothness, connection, integrality, and reducibility. ☐

We now want to study some geometrical properties of the stack $\mathrm{D}(M)$-Cov and, therefore, of the equivariant Hilbert schemes.

Remark 4.11. The ring R_M can be written as the quotient of the ring $\mathbb{Z}[x_{m,n}]_{(m,n)\in J}$, where J is $\{(m,n) \in M^2 \mid m, n, m+n \neq 0\}$ divided by the equivalence relation $(m,n) \sim (n,m)$, by the ideal

$$I = \begin{pmatrix} x_{m,n}x_{m+n,t} - x_{n,t}x_{n+t,m} \text{ with } m, n, t, m+n, n+t, m+n+t \neq 0 \text{ and } m \neq t, \\ x_{-m,t}x_{-m+t,m} - x_{-m,s}x_{-m+s,m} \text{ with } m, s, t \neq 0 \text{ and distinct} \end{pmatrix}.$$

Indeed, the first relations are trivial when one of m, n, and t is zero or $m = t$, while if $m + n = 0$ yield relations $x_{m,-m} = x_{-m,t}x_{-m+t,m}$. Using these last relations, we can remove all the variables $x_{m,n}$ with $0 \in \{m, n, m+n\}$. ☐

Remark 4.12. There exists a map $f : \tilde{K}_+ \longrightarrow \mathbb{N}$ such that for any $m, n \neq 0$ we have $f(e_{m,n}) = 1$ if $m + n \neq 0$, $f(e_{m,-m}) = 2$ otherwise. In particular, $f(v) = 0$ only if $v = 0$. Moreover, f induces an $\mathbb{N}$-graduation on both $(R_M \otimes A)$ and $\mathbb{Z}[K_+] \otimes A$, where A is a ring, such that the degree zero part is A and that the elements $x_{m,n}$ with $m + n \neq 0$ are homogeneous of degree 1. The map f is obtained as the composition $\tilde{K}_+ \longrightarrow K \subseteq \mathbb{Z}^M/\langle e_0\rangle \xrightarrow{h} \mathbb{Z}$, where $h(e_m) = 1$ if $m \neq 0$. ☐

One of the open problems in the theory of equivariant Hilbert schemes is whether those schemes are connected. As said above $M\text{-Hilb}^{\underline{m}}$ is connected if and only if $\mathrm{D}(M)\text{-Cov}^{\underline{m}}$ is so. What we can say here is:

Theorem 4.13. The stack $\mathrm{D}(M)$-Cov is connected with geometrically connected fibers. If $M - \{0\} \subseteq \underline{m}$, then M-$\mathrm{Hilb}^{\underline{m}}$ has the same properties. □

Proof. It is enough to prove that $\operatorname{Spec} R_M \otimes k$ is connected for any field k. But $R_M \otimes k$ has an $\mathbb{N}$-graduation such that $(R_M \otimes k)_0 = k$ by Remark 4.12 and it is a general fact that such an algebra does not contain nontrivial idempotents. ■

We now want to discuss the problem of the reducibility of $\mathrm{D}(M)$-Cov.

Definition 4.14. Let S be a scheme. An S-scheme X is said *universally reducible* over S if, for any base change $S' \longrightarrow S$, the scheme $X \times_S S'$ is reducible. A scheme is universally reducible if it is so over $\mathbb{Z}$. □

Remark 4.15. It is easy to check that X is universally reducible over S if and only if all the fibers are reducible. □

Lemma 4.16. If there exist $m, n, t, a \in M$ such that

(1) m, n, and t are distinct and not zero;
(2) $a \neq 0, m, n, t, m-n, n-m, n-t, t-n, m-t, 2m-t, 2n-t, m+n-t, m+n-2t$;
(3) $2a \neq m+n-t$;

then $\operatorname{Spec} R_M$ is universally reducible. □

Proof. Let k be a field and $I = (\underline{x}^{\alpha_i} - \underline{x}^{\beta_i})$ be an ideal of $k[x_1, \dots, x_r] = k[\underline{x}]$. We will say that $\alpha \in \mathbb{N}^r$ is transformable (with respect to I) if there exists i such that $\alpha_i \leq \alpha$ or $\beta_i \leq \alpha$. Here, by $\alpha \leq \beta \in \mathbb{N}^r$ we mean $\alpha_j \leq \beta_j$ for all j. A direct computation shows that if $\underline{x}^{\alpha} - x^{\beta} \in I$ and $\alpha \neq \beta$, then both α and β are transformable.

We will use the above notation for the ideal I defining $R_M \otimes k$ as in Remark 4.11. In particular, the elements $\alpha_i, \beta_i \in \mathbb{N}^J$ associated to the ideal I are of the form $e_{u,v} + e_{u+v,w}$ with $u, v, u+v, w, u+v+w \neq 0$.

Set $\mu = \prod_{m,n} x_{m,n}$. Since $R_M \otimes k \longrightarrow k[K_+] \subseteq k[K] = (R_M \otimes k)_\mu$, there exists $N > 0$ such that $P = \operatorname{Ker}(R_M \otimes k \longrightarrow k[K_+]) = \operatorname{Ann} \mu^N$. Our strategy will be to find an element of P which is not nilpotent. Since P is a minimal prime, being $\operatorname{Spec} k[K_+]$ an irreducible component of $\operatorname{Spec} R_M \otimes k$, it follows that $R_M \otimes k$ is reducible. Now consider $\alpha = e_{a,m-a} + e_{m+n-t-a,t+a-m} + e_{t+a-n,n-a}$, $\beta = e_{m+n-t-a,t+a-n} + e_{a,n-a} + e_{m-a,t+a-m} \in \mathbb{N}^J$ and $z = \underline{x}^{\alpha} - \underline{x}^{\beta}$. We will show that $\mu z = 0$, that is, $z \in P$ and that z is not nilpotent. First of all note that z is well defined since for any $e_{u,v}$ in α or β we have $u, v \neq 0$ and $0 \neq u + v \in \{m, n, t\}$ thanks to (1) and (2). Let S_M be the universal algebra over R_M, that is, $S_M = \bigoplus_{m \in M} R_M v_m$ with $v_m v_n = x_{m,n} v_{m+n}$ and $v_0 = 1$. By construction, we have

$$(v_a v_{m-a})(v_{m+n-t-a} v_{t+a-m})(v_{t+a-n} v_{n-a}) = \underline{x}^{\alpha} v_m v_n v_t$$

$$= (v_{m+n-t-a} v_{t+a-n})(v_a v_{n-a})(v_{m-a,t+a-m}) = \underline{x}^{\beta} v_m v_n v_t$$

So $\underline{x}^{\alpha} x_{m,n} x_{m+n,t} v_{m+n+t} = \underline{x}^{\beta} x_{m,n} x_{m+n,t} v_{m+n+t}$ and therefore $z\mu = 0$, that is, $z \in P$.

Now, we want to prove that any linear combination $\gamma = a\alpha + b\beta \in \mathbb{N}^J$ with $a, b \in \mathbb{N}$ is not transformable. First remember that each $e_{u,v}$ in γ is such that $u + v \in \{m, n, t\}$. If we have $e_{u,v} + e_{u+v,w} \leq \gamma$, then there must exist $e_{i,j} \leq \gamma$ such that $i \in \{m, n, t\}$ or $j \in \{m, n, t\}$. Condition (2) is exactly what we need to avoid this situation and can be written as $\{a, m - a, m + n - t - a, t + a - m, t + a - n, n - a\} \cap \{m, n, t\} = \emptyset$.

In particular, if we think of $\tilde{K}_+$ as a quotient of $\mathbb{N}^J$, we have $a\alpha + b\beta = a'\alpha + b'\beta$ in $\tilde{K}_+$ if and only if they are equal in $\mathbb{N}^J$. Assume for a moment that $\alpha \neq \beta$ in $\mathbb{N}^J$. Clearly, this means that α and β are $\mathbb{Z}$-independent in $\mathbb{Z}^J$. Since any linear combination of α and β is not transformable, it follows that $\underline{x}^{\alpha}$ and $\underline{x}^{\beta}$ are algebraically independent over k in $R_M \otimes k$ and, in particular, that $z = \underline{x}^{\alpha} - \underline{x}^{\beta}$ cannot be nilpotent. So it remains to prove that $\alpha \neq \beta$ in $\mathbb{N}^J$. Note that for any $i \in \{m, n, t\}$, there exists only one $e_{u,v}$ in α such that $u + v = i$ and the same happens for β. So, if $\alpha = \beta$ and since m, n, and t are distinct, those terms have to be equal, for instance, $e_{a,m-a} = e_{m+n-t-a,t+a-n}$. But $a \neq m + n - t - a$ by (3), while $a \neq t + a - n$ since $t \neq n$. Therefore, $\alpha \neq \beta$. ■

Corollary 4.17. If $|M| > 7$ and $M \not\simeq (\mathbb{Z}/2\mathbb{Z})^3$ then D(M)-Cov is universally reducible and the same holds for M-Hilb$^{\underline{m}}$ provided that $\underline{m}$ contains all elements of $M - \{0\}$. □

Proof. We have to show that R_M is universally reducible and so we will apply Lemma 4.16. If $M = C \times T$, where C is cyclic with $|C| \geq 4$ and $T \neq 0$ we can choose: m a generator of C, $n = 3m$, $t = 2m$ and $a \in T - \{0\}$. If M cannot be written as above, there are four remaining cases. (1) $M \simeq \mathbb{Z}/8\mathbb{Z}$: choose $m = 2$, $n = 4$, $t = 6$, $a = 1$. (2) M cyclic with $|M| > 8$ and $|M| \neq 10$: choose $m = 1$, $n = 2$, $t = 3$, $a = 5$. (3) $M \simeq (\mathbb{Z}/2\mathbb{Z})^l$ with $l \geq 4$: choose $m = e_1$, $n = e_2$, $t = e_3$, $a = e_4$. (4) $M \simeq (\mathbb{Z}/3\mathbb{Z})^l$ with $l \geq 2$: choose $m = e_1$, $n = 2e_1$, $t = e_2$, $a = m + t = e_1 + e_2$. ■

Proposition 4.18. D(M)-Cov is smooth if and only if $\mathcal{Z}_M$ is so. This happens if and only if $M \simeq \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in these cases D(M)-Cov $= \mathcal{Z}_M$. To be more precise, $R_M = \mathbb{Z}[x_{m,n}]_{(m,n)\in J}$, where J is the set defined in Remark 4.11.

In particular, M-Hilb$^{\underline{m}}$ is smooth and irreducible for any sequence $\underline{m}$ if M is as above. Otherwise, if $M - \{0\} \subseteq \underline{m}$, M-Hilb$^{\underline{m}}$ is not smooth. □

Proof. Let k be a field. Note that

$$\mathrm{D}(M)\text{-Cov smooth} \iff R_M \text{ smooth} \Longrightarrow \mathcal{Z}_M \text{ smooth } \Longrightarrow k[K_+]/k \text{ smooth.}$$

We first prove that if $k[K_+]$ is smooth then M has to be one of the groups of the statement. We have $K_+ \simeq \mathbb{N}^r \oplus \mathbb{Z}^s$ and therefore $k[K_+]$ is UFD. We will consider $k[K_+]$ endowed with the $\mathbb{N}$-graduation defined in Remark 4.12. Since any of the $x_{m,n}$ has degree 1, it is irreducible and so prime. If we have a relation $x_{m,n}x_{m+n,t} = x_{n,t}x_{n+t,m}$ with $m, n, t, m+n, n+t, m+n+t \neq 0$ and $m \neq t$, then $x_{m,n} \mid x_{n,t}x_{n+t,m}$ implies that $x_{m,n} = x_{n,t}$ or $x_{m,n} = x_{n+t,m}$, which is impossible thanks to our assumptions. We will prove that if M is not isomorphic to one of the group in the statement, then such a relation exists. Clearly, it is enough to find this relation in a subgroup of M. So it is enough to consider the following cases. (1) M cyclic with $|M| \geq 5$: choose $m = n = 1, t = 2$. (2) $M \simeq \mathbb{Z}/4\mathbb{Z}$: choose $m = 1, n = 2, t = 3$. (3) $M \simeq (\mathbb{Z}/2\mathbb{Z})^3$: choose $m = e_1, n = e_2, t = e_3$. (4) $M \simeq (\mathbb{Z}/3\mathbb{Z})^2$: choose $m = n = e_1, t = e_2$.

We now want to prove that when M is as in the statement, then the ideal I of Remark 4.11 is zero. If we have a relation as in the first row, since $m \neq t$ we have $|M| \geq 3$. If $M \simeq \mathbb{Z}/3\mathbb{Z}$ then $t = 2m$ and $m + t = 0$. If $M \simeq (\mathbb{Z}/2\mathbb{Z})^2$, if m, n, and t are distinct then $m + n + t = 0$, otherwise $m = n$ and $m + n = 0$. If we have a relation as in the second row, since m, t, and s are distinct, we must have $M \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Therefore, $m + t = s$ and the relation becomes trivial. ■

Corollary 4.19. *D*$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$*-Cov is isomorphic to the stack of sequences* $(\mathcal{L}_i, \psi_i)_{i=1,2,3}$*, where* $\mathcal{L}_1, \mathcal{L}_2$*, and* $\mathcal{L}_3$ *are invertible sheaves and* $\psi_1 : \mathcal{L}_2 \otimes \mathcal{L}_3 \longrightarrow \mathcal{L}_1$, $\psi_2 : \mathcal{L}_1 \otimes \mathcal{L}_3 \longrightarrow \mathcal{L}_2$, $\psi_3 : \mathcal{L}_1 \otimes \mathcal{L}_2 \longrightarrow \mathcal{L}_3$ *are maps.* □

Proof. Set $M = (\mathbb{Z}/2\mathbb{Z})^2$. Thanks to Proposition 4.18, we know that $\tilde{K}_+ = K_+ \simeq \mathbb{N}v_{e_1,e_2} \oplus \mathbb{N}v_{e_1,e_1+e_2} \oplus \mathbb{N}v_{e_2,e_1+e_2}$. So an object of D($M$)-Cov is given by invertible sheaves $\mathcal{L}_1 = \mathcal{L}_{e_1}, \mathcal{L}_2 = \mathcal{L}_{e_2}, \mathcal{L}_3 = \mathcal{L}_{e_1+e_2}$ and maps $\psi_1 = \psi_{e_2,e_1+e_2}, \psi_2 = \psi_{e_1,e_1+e_2}, \psi_3 = \psi_{e_1,e_2}$. ■

Remark 4.20. $\mathrm{D}(\mathbb{Z}/4\mathbb{Z})$-Cov and $\mathbb{Z}/4\mathbb{Z}$-Hilb$^{\underline{m}}$, for any sequence $\underline{m}$, are integral and normal since one can check directly that $R_{\mathbb{Z}/4\mathbb{Z}} = \mathbb{Z}[x_{1,2}, x_{3,3}, x_{2,3}, x_{1,1}]/(x_{1,2}x_{3,3} - x_{2,3}x_{1,1})$. I am not able to prove that $\mathrm{D}(M)$-Cov is irreducible when M is one of $\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/7\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^3$. Anyway the first two cases seem to be integral thanks to a computer program, while for the last ones there are some techniques that can be used to study this problem but they are too complicated to be explained here. □

4.2 The invariant $h\colon |\mathrm{D}(M)\text{-Cov}| \longrightarrow \mathbb{N}$

In this subsection, we investigate the local structure of a $\mathrm{D}(M)$-cover, especially over a local ring. In particular, we will define an upper semicontinuous map $h\colon |\mathrm{D}(M)\text{-Cov}| \longrightarrow \mathbb{N}$ that measures how much a cover fails to be a torsor: the open locus $\mathrm{BD}(M) \subseteq \mathrm{D}(M)$-Cov will exactly be the locus $\{h = 0\}$.

Notation 4.21. Given a ring A, we will write $B \in \operatorname{Spec} R_M(A)$ meaning that B is an M-graded A-algebra with a given M-graded basis, usually denoted by $\{v_m\}_{m\in M}$ with $v_0 = 1$, and a given multiplication ψ such that

$$B = \bigoplus_{m\in M} Av_m, \qquad v_m v_n = \psi_{m,n} v_{m+n}.$$

We will also denote by A^* the group of invertible elements of A. If $f\colon X \longrightarrow Y$ is an affine map of schemes and $q \in Y$, we will use the notation $\mathcal{O}_{X,q} = f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y,q}$. In particular, $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,q} \simeq \operatorname{Spec} \mathcal{O}_{X,q}$. Note that, although $\mathcal{O}_{X,q}$ is written as a localization in a point, this ring is not local in general. □

Lemma 4.22. Let A be a ring and $B \in \operatorname{Spec} R_M(A)$, with graded basis v_m and multiplication map ψ. Then the set

$$H_\psi = H_{B/A} = \{m \in M \mid v_m \in B^*\} = \{m \in M \mid \psi_{m,-m} \in A^*\}$$

is a subgroup of M. Moreover, if $m, n \in M$ and $h \in H_\psi$ then $\psi_{m,n}$ and $\psi_{m,n+h}$ differs by an element of A^*. If H is a subgroup of H_ψ then $C = \bigoplus_{m\in H} Av_m$ is an element of $\mathrm{BD}(H(A))$. Moreover if $\sigma : M/H \longrightarrow M$ gives representatives of M/H in M and we set $w_m = v_{\sigma(m)}$ for $m \in M/H$ we have

$$B = \bigoplus_{m\in M/H} Cw_m \in \operatorname{Spec} R_{M/H}(C).$$

Finally, if we denote by ψ' the induced multiplication on B over C we have $H_{\psi'} = H_\psi / H$ and for any $m, n \in M$ $\psi'_{m,n}$ and $\psi_{m,n}$ differ by an element of C^*. □

Proof. From the relations $v_m v_{-m} = \psi_{m,-m}$, $v_m^{|M|-1} = \lambda v_{-m}$, $v_m^{|M|} = \lambda \psi_{m,-m}$, where $\lambda \in B$ and $v_m v_n = \psi_{m,n} v_{m+n}$ we see that $v_m \in B^* \iff \psi_{m,-m} \in A^*$ and that $H_\psi < M$. From (4.1), we get the relations $\psi_{-h,h} = \psi_{h,u}\psi_{h+u,-h}$ and $\psi_{m,n}\psi_{m+n,h} = \psi_{n,h}\psi_{m,n+h}$. So if $h \in H$, then $\psi_{h,u} \in A^*$ for any u and $\psi_{m,n}$ and $\psi_{m,n+h}$ differ by an element of A^*.

Now consider the second part of the statement. From Proposition 4.3, we know that C is a torsor over A. Since for any m we have $v_m = (\psi_{h,m}/v_h)v_{\sigma(\bar{m})}$, where $h = \sigma(\bar{m}) - m \in H$ we obtain the expression of B as M/H graded C-algebra and that

$$\psi'_{m,n} = \psi_{\sigma(m),\sigma(n)}(\psi_{h,\sigma(m)+\sigma(n)}/v_h) \quad \text{where } h = \sigma(m+n) - \sigma(m) - \sigma(n).$$

From the above equation, it is easy to conclude the proof. ■

Definition 4.23. Given a ring A and $B \in \operatorname{Spec} R_M(A)$, we continue to use the notation $H_{B/A}$ introduced in Lemma 4.22 and we will call the algebra C obtained for $H = H_{B/A}$ the *maximal torsor* of the extension B/A. If k is a field and $\mathcal{E} \in K_+^\vee$ we will write $H_\mathcal{E} = H_{B/k}$, where B is the algebra induced by the multiplication $0^\mathcal{E}$. In particular,

$$H_\mathcal{E} = \{m \in M \mid \mathcal{E}_{m,-m} = 0\}.$$

Finally, if $f : X \longrightarrow Y \in \mathrm{D}(M)\text{-Cov}(Y)$ and $q \in Y$ we define $\mathcal{H}_f(q) = H_{\mathcal{O}_{X,q}/\mathcal{O}_{Y,q}}$. □

Proposition 4.24. We have a map

$$\begin{tikzcd}
|\mathrm{D}(M)\text{-Cov}| \arrow[r, "\mathcal{H}"] & \{\text{subgroups of } M\} \\
B/k \arrow[r, mapsto] & H_{B/k}
\end{tikzcd}$$

such that, if $Y \xrightarrow{u} \mathrm{D}(M)\text{-Cov}$ is given by $X \xrightarrow{f} Y$, then $\mathcal{H}_f = \mathcal{H} \circ |u|$. □

Proof. It is enough to note that if A is a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ is given by multiplications ψ and $\pi : A \longrightarrow A/m_A \longrightarrow k$ is a morphism, where k is a field, then $\psi_{m,-m} \in A^* \iff \pi(\psi_{m,-m}) \neq 0$. ■

Remark 4.25. Let (A, m_A) be a local ring and $B \in \operatorname{Spec} R_M(A)$ with M-graded basis $\{v_m\}_{m\in M}$. Then $H_{B/A} = \mathcal{H}_{B/A}(m_A)$. If $H_{B/A} = 0$ then any v_m, with $m \neq 0$, is nilpotent in $B \otimes k$ and therefore B is local with maximal ideal

$$m_B = m_A \oplus \bigoplus_{m\in M-\{0\}} Av_m$$

and residue field $B/m_B = A/m_A$. In particular, m_B/m_B^2 is M-graded. □

Lemma 4.26. Let A be a local ring and $B = \bigoplus_{m\in M} Av_m \in \mathrm{D}(M)\text{-Cov}(A)$ such that $H_{B/A} = 0$. If $m_1, \dots, m_r \in M$ then B is generated in degrees $m_1, \dots, m_r$ as an A-algebra if and only if $m_B = (m_A, v_{m_1}, \dots, v_{m_r})_B$. □

Proof. We can write $m_B = m_A \oplus \bigoplus_{m\in M-\{0\}} Av_m$. Denote $\underline{v} = v_{m_1}, \dots, v_{m_r}$ and $\pi(\alpha) = \sum_i \alpha_i m_i$ for $\alpha \in \mathbb{N}^r$. The "only if" follows since given $l \in M - \{0\}$ there exists a relation of the form $v_l = \mu \underline{v}^\alpha$ with $\mu \in A^*$ and $\alpha \neq 0$ and so $v_l \in (m_A, v_{m_1}, \dots, v_{m_r})_B$. For the converse note that, given $l \in M - \{0\}$, $v_l \in m_B = (m_A, v_{m_1}, \dots, v_{m_r})$ means that we have a relation $v_l = \lambda v_{l'} v_{m_i}$ for some i, $\lambda \in A^*$ and $l' = l - m_i$. Moreover, $v_l \notin A[\underline{v}]$ implies that $v_{l'} \notin A[\underline{v}]$ and $l' \neq 0$. If, by contradiction, we have such an element l we can write $v_l = \mu v_{n_1} \cdots v_{n_s}$ with $n_i \in M - \{0\}$ and $s \geq |M|^2$. In particular, there must exist i such that $m = n_i$ appears at least $|M|$ times in this product. So $m_A \ni v_m^{|M|} \mid v_l$ and $v_l \in m_A B$, which is not the case. ■

Assume that we have a cover $X \xrightarrow{f} Y \in \mathrm{D}(M)\text{-Cov}(Y)$. We want to define, for any $m \in M$ a map $h_{f,m} = h_{X/Y,m} : Y \longrightarrow \{0, 1\}$. Let $q \in Y$ and denote by C the "maximal torsor" of $\mathcal{O}_{X,q}/\mathcal{O}_{Y,q}$ (see Definition 4.23). Also let $p \in f^{-1}(q)$ and set $p_C = p \cap C$. Taking into account Remark 4.25, we know that $B = (\mathcal{O}_{X,q})_p = (\mathcal{O}_{X,q})_{p_C}$ and that $B \in \mathrm{D}(M/\mathcal{H}_f(q))\text{-Cov}(C_{p_C})$ with $H_{B/C_{p_C}} = 0$. Moreover, B is local, $B/m_B = C_{p_C}/p_C$ and m_B/m_B^2 is $(M/\mathcal{H}_f(q))$-graded. If we denote by $\bar{m}$ the image of $m \in M$ in $M/\mathcal{H}_f(q)$ and by $(m_B/m_B^2)_t$ the graded pieces of m_B/m_B^2, where $t \in M/\mathcal{H}_f(q)$, we can define:

Definition 4.27. With the above notation, we set

$$h_{f,m}(q) = \begin{cases} 0 & \text{if } m \in \mathcal{H}_f(q), \\ \dim_{C_{p_C}/p_C} (m_B/m_B^2)_{\bar{m}} & \text{otherwise.} \end{cases}$$

We also set

$$h_f(q)=\dim_{C_{p_C}/p_C}(m_B/m_B^2)-\dim_{C_{p_C}/p_C}(m_B/m_B^2)_0=\left(\sum_{m\in M}h_{f,m}(q)\right)/|\mathcal{H}_f(q)|.$$

If $\mathcal{E}\in K_+^\vee$ we set $h_{\mathcal{E},m}=h_{f,m}$, $h_{\mathcal{E}}=h_f\in\mathbb{N}$ where f is the cover $\operatorname{Spec} A\longrightarrow\operatorname{Spec} k$ and A is the algebra given by multiplication $0^{\mathcal{E}}$ over some field k. □

The following lemma shows that the value of $h_{f,m}(q)$ does not depend on the choice of the point $p\in X$ over $q\in Y$.

Lemma 4.28. Let (A,m_A) be a local ring, $B\in \mathrm{D}(M)\text{-}\mathrm{Cov}(A)$ given by the multiplication ψ and $t\in M$. Set also $h_{B/A,t}=h_{B/A,t}(m_A)$, for some choice of a prime of B over m_A. Then $h_{B/A,t}=1$ if and only if the following conditions are satisfied:

- $t\notin H_{B/A}$;
- for all $u,n\in M-H_{B/A}$ such that $u+n\equiv t \bmod H_{B/A}$ we have $\psi_{u,n}\notin A^*$. □

Proof. Let C be the maximal torsor of the extension B/A and p be a maximal prime of B. We use notation from Lemma 4.22. For any $l\in M-H_{B/A}$, we have a surjective map

$$k(p)=(m_{B_p}/pC_p)_{\bar{l}}\longrightarrow(m_{B_p}/m_{B_p}^2)_{\bar{l}}$$

and so $\dim_{k(p)}(m_{B_p}/m_{B_p}^2)_{\bar{l}}\in\{0,1\}$, where $\bar{l}$ is the image of l under the projection $M\longrightarrow M/H_{A/B}$. If we prove the last part of the statement clearly we will also have that $h_{B/A,t}$ is well defined. If $t\in H_{B/A}$ then $h_{B/A,t}=0$, while if there exist u,n as in the statement such that $\psi_{u,n}\in A^*$, then $w_{\bar{t}}\in C_p^*w_{\bar{u}}w_{\bar{n}}\subseteq m_{B_p}^2$ and again $h_{B/A,t}=0$. On the other hand, if $h_{B/A,t}=0$ and $t\notin H_{B/A}$ then $w_{\bar{t}}\in m_{B_p}^2$ and therefore we have an expression

$$w_{\bar{t}}=bx+\sum_{\bar{u},\bar{n}\neq 0}b_{\bar{u},\bar{n}}w_{\bar{u}}w_{\bar{n}}\quad\text{with } b,b_{\bar{u},\bar{n}}\in B_p,\ x\in m_{C_p}$$

The second sum splits as a sum of products of the form $c_{s,\bar{u},\bar{n}}w_sw_{\bar{u}}w_{\bar{n}}$ with $s+\bar{u}+\bar{n}=\bar{t}$ and $c_{s,\bar{u},\bar{n}}\in C_p$. Since C_p is local, one of these monomials generates $C_pw_{\bar{t}}$. In this case, if $s+\bar{u}=0$ then $\bar{u}\in H_{B_p/C_p}=0$ which is not the case. So we have an expression

$$w_{\bar{t}}=\lambda w_{\bar{u}}w_{\bar{n}}=\lambda\psi'_{\bar{u},\bar{n}}w_{\bar{t}}\Longrightarrow\psi'_{\bar{u},\bar{n}}\in C_p^*,$$

where $\bar{u}, \bar{n} \neq 0$ and $\bar{u} + \bar{n} = \bar{t}$. Since $\psi'_{\bar{u},\bar{n}}$ and $\psi_{u,n}$ differs by an element of C^* thanks to Lemma 4.22, it follows that $\psi_{u,n} \in A^*$. ■

Proposition 4.29. We have maps

$$\begin{tikzcd} {|\mathrm{D}(M)\text{-Cov}|} \arrow[r, "h_m"] & \{0, 1\} & {|\mathrm{D}(M)\text{-Cov}|} \arrow[r, "h"] & \mathbb{N} \\ B/k \arrow[r, mapsto] & h_{B/k,m} & B/k \arrow[r, mapsto] & h_{B/k} \end{tikzcd}$$

such that, if $Y \xrightarrow{u} \mathrm{D}(M)\text{-Cov}$ is given by $X \xrightarrow{f} Y$, then $h_{f,m} = h_m \circ |u|$ and $h_f = h \circ |u|$. □

Proof. Taking into account Lemma 4.28 and Proposition 4.24, it is enough to note that if A is a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ is given by multiplications ψ and $\pi : A \longrightarrow A/m_A \longrightarrow k$ is a morphism, where k is a field, then $\psi_{u,v} \in A^* \iff \pi(\psi_{u,v}) \neq 0$ and $H_{B/A} = H_{B \otimes_A k/k}$. ■

Corollary 4.30. Under the hypothesis of Lemma 4.26, $\{m \in M \mid h_{B/A,m} = 1\}$ is the minimum among the subsets Ω of M such that B is generated as an A-algebra in the degrees Ω. In particular, B is generated in $h_{B/A}$ degrees. □

Proposition 4.31. Let (A, m_A) be a local ring, $B \in \mathrm{D}(M)\text{-Cov}(A)$ and C the maximal torsor of B/A. Then

$$h_{B/A}(m_A) = \dim_{k(p)} \Omega_{B/C} \otimes_B k(p)$$

for any maximal prime p of B. In particular, if $(|H_{B/A}|, \mathrm{char}\, A/m_A) = 1$ we also have $h_{B/A}(m_A) = \dim_{k(p)} \Omega_{B/A} \otimes_B k(p)$ for any maximal prime p of B. □

Proof. If A is any ring and $B \in \mathrm{D}(M)\text{-Cov}(A)$ is given by basis $\{v_m\}_{m \in M}$ and multiplication ψ one sees from the universal property that

$$\Omega_{B/A} = B^M / \langle e_0, v_n e_m + v_m e_n - \psi_{m,n} e_{m+n} \rangle.$$

Now consider $B \in \mathrm{D}(M/H)\text{-Cov}(C)$, where $H = H_{B/A}$ and let p be a maximal prime of B. Following the notation of Lemma 4.22, we have that $w_m \in p$ for any $m \in M/H - \{0\}$ and $\psi'_{m,n} \in p \iff \psi_{m,n} \in m_A$. So $\Omega_{B/C} \otimes_B k(p)$ is free on the e_m for $m \in M/H - \{0\}$ such that for any $u, n \in M/H - \{0\}$, $u + n = m$ implies $\psi_{u,n} \notin A^*$, that are exactly $h_{B/A}(m_A)$ thanks to Lemma 4.28. ■

Corollary 4.32. The function h is upper semicontinuous. ☐

Proof. Let $X \xrightarrow{f} Y$ be a $\mathrm{D}(M)$-cover and $q \in Y$. Set $r = h_f(q)$ and $H = \mathcal{H}_f(q)$. We can assume that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ with graded basis $\{v_m\}_{m \in M}$ and multiplication ψ and that $\psi_{m,-m} \in A^*$ for any $m \in H$. Set $C = A[v_m]_{m \in H}$. The ring C_q is the maximal torsor of B_q/A_q and so, if $p \in X$ is a point over q, we have $r = \dim_{k(p)} \Omega_{B/C} \otimes_B k(p)$. Finally, let $U \subseteq X$ be an open neighborhood of p such that $\dim_{k(p')} \Omega_{B/C} \otimes_B k(p') \leq r$ for any $p' \in U$ and $V = f(U)$. We want to prove that $h \leq r$ on V. Indeed given $q' = f(p') \in V$, if D is the maximal torsor of $B_{q'}/A_{q'}$, we have $C_{q'} \subseteq D \subseteq B_{q'}$. So

$$h_f(q') = \dim_{k(p')} \Omega_{B_{q'}/D} \otimes_{B_{q'}} k(p') \leq \dim_{k(p')} \Omega_{B_{q'}/C_{q'}} \otimes_{B_{q'}} k(p') \leq r. \qquad \blacksquare$$

Remark 4.33. The 0 section $R_M \longrightarrow \mathbb{Z}$, that is, the map that sends any $x_{m,n}$ with $m, n \neq 0$ to zero, induces a closed immersion

$$\underline{\mathrm{Pic}}^{|M|-1} \simeq \mathrm{B}\mathcal{T} = [\operatorname{Spec} \mathbb{Z}/\mathcal{T}] \subseteq [\operatorname{Spec} R_M/\mathcal{T}] \simeq \mathrm{D}(M)\text{-Cov},$$

where $\mathcal{T} = \mathrm{D}(\mathbb{Z}^M/\langle e_0 \rangle)$. ☐

Proposition 4.34. The following results hold:

(1) $\{h = 0\} = |\mathrm{BD}(M)|$;
(2) $\{h \geq |M|\} = \emptyset$;
(3) $\{h = |M| - 1\} = |\mathrm{BD}(\mathbb{Z}^M/\langle e_0 \rangle)|$ (see Remark 4.33). ☐

Proof. If $X \xrightarrow{f} Y$ is a $\mathrm{D}(M)$-torsor, clearly $h_f = 0$. So (1) and (2) follow from Corollary 4.30. Finally, if $B \in \mathrm{D}(M)\text{-Cov}(k)$ with multiplication ψ, $h_{B/k} = |M| - 1$ if and only if $H_{B/k} = 0$ and $h_{B/k,m} = 1$ for all $m \in M - \{0\}$. This means that $\psi_{m,n} = 0$ for any $m, n \neq 0$ by Lemma 4.28. ■

In particular, setting $U_i = \{h \leq i\}$, we obtain a stratification $\mathrm{BD}(M) = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{|M|-1} = \mathrm{D}(M)\text{-Cov}$ of $\mathrm{D}(M)$-Cov by open substacks.

4.3 The locus $h \leq 1$

In this subsection, we want to describe $\mathrm{D}(M)$-covers with $h \leq 1$. This means that "up to torsors" we have a graded M-algebra generated over the base ring in one degree. We will see that $\{h \leq 1\}$ is a smooth open substack of $\mathcal{Z}_M$ determined by a special class of explicit smooth extremal rays of K_+. This will allow us to give a description of covers over locally noetherian and locally factorial scheme X with $(\operatorname{char} X, |M|) = 1$ whose total space is normal. Such a description, when X is a smooth algebraic variety over an algebraic closed field k with $(\operatorname{char} k, |M|) = 1$, was already given in [18, Theorem 2.1, Corollary 3.1].

Notation 4.35. Given $\mathcal{E} \in K_+^\vee$ we will write $\mathcal{E}_{m,n} = \mathcal{E}(v_{m,n})$. Since $K \otimes \mathbb{Q} \simeq \mathbb{Q}^M/\langle e_0 \rangle$ we will also write $\mathcal{E}_m = \mathcal{E}(e_m) \in \mathbb{Q}$, so that $\mathcal{E}_{m,n} = \mathcal{E}_m + \mathcal{E}_n - \mathcal{E}_{m+n}$. When we will have to consider different abelian groups, we will write K_{+M}, K_M instead of, respectively, K_+, K, in order to avoid confusion. Given a group homomorphism $\eta : M \longrightarrow N$, we will denote by $\eta_* : K_M \longrightarrow K_N$ the homomorphism such that $\eta_*(v_{m,n}) = v_{\eta(m),\eta(n)}$ for all $m, n \in M$, where K_M is the group associated to K_+, □

Remark 4.36. Let A be a ring and consider a sequence $\underline{\mathcal{E}} = \mathcal{E}^1, \dots, \mathcal{E}^r \in K_+^\vee$. An element of $\mathcal{F}_{\underline{\mathcal{E}}}(A)$ coming from the atlas (see Remark 3.13) is given by a pair $(\underline{z}, \lambda)$ where $\underline{z} = z_1, \dots, z_r \in A$ and $\lambda : K \longrightarrow A^*$. The image of this object under $\pi_{\underline{\mathcal{E}}}$ is the algebra whose multiplication is given by $\psi_{m,n} = \lambda_{m,n}^{-1} z_1^{\mathcal{E}^1_{m,n}} \cdots z_r^{\mathcal{E}^r_{m,n}}$. □

Lemma 4.37. Let $\eta : M \longrightarrow N$ be a surjective morphism and $\underline{\mathcal{E}}$ be a sequence in $(K_{+N})^\vee$. Then $\underline{\mathcal{E}}$ is a smooth sequence for N if and only if $\underline{\mathcal{E}} \circ \eta_*$ is a smooth sequence for M. □

Proof. We want to apply Lemma 3.38. Therefore, we have to prove that $\eta_*(K_{+M}) = K_{+N}$, which is clear, and that $\operatorname{Ker} \eta_* = \langle \operatorname{Ker} \eta_* \cap K_{+N} \rangle$. Consider the map $f : \mathbb{Z}^M/\langle e_0 \rangle \longrightarrow \mathbb{Z}^N/\langle e_0 \rangle$ given by $f(e_m) = e_{\eta(m)}$ and set $H = \operatorname{Ker} \eta$. Clearly, $f_{|K_M} = \eta_*$. It is easy to check that $G = \langle v_{m,n} \text{ for } m \in H \rangle_{\mathbb{Z}} \subseteq \operatorname{Ker} \eta^* \subseteq \operatorname{Ker} f$ and that $\operatorname{Ker} f / \operatorname{Ker} \eta_* \simeq H$. So in order to conclude, it is enough to note that the map $H \longrightarrow \operatorname{Ker} f/G$ sending h to e_h is a surjective group homomorphism since we have relations $e_h + e_{h'} - e_{h+h'} = v_{h,h'}$ and $e_{m+h} - e_m = e_h - v_{m,h}$ for $m \in M$ and $h, h' \in H$. ■

Proposition 4.38. Let $\eta : M \longrightarrow \mathbb{Z}/l\mathbb{Z}$ be a surjective homomorphism with $l > 1$. Then

$$\mathcal{E}^\eta(v_{m,n}) = \begin{cases} 0 & \text{if } \eta(m) + \eta(n) < l, \\ 1 & \text{otherwise} \end{cases}$$

defines a smooth extremal ray for K_+. □

Proof. $\mathcal{E}^\eta \in K_+^\vee$ because, if $\sigma : \mathbb{Z}/l\mathbb{Z} \longrightarrow \mathbb{N}$ is the obvious section, $\mathcal{E}^\eta$ is the restriction of the map $\mathbb{Z}^M/\langle e_0\rangle \longrightarrow \mathbb{Z}$ sending e_m to $\sigma(\eta(m))$. In order to conclude the proof, we will apply Lemma 4.37 and Proposition 3.37. Set $N = \mathbb{Z}/l\mathbb{Z}$. One clearly has $\mathcal{E}^\eta = \mathcal{E}^{\mathrm{id}} \circ \eta_*$ and so we can assume $M = \mathbb{Z}/l\mathbb{Z}$ and $\eta = \mathrm{id}$. In this case, one can check that $v_{1,1}, v_{1,2}, \dots, v_{1,l-1}$ is a $\mathbb{Z}$-base of K such that $\mathcal{E}^\eta(v_{1,j}) = 0$ if $j < l-1$, $\mathcal{E}^\eta(v_{1,l-1}) = 1$. ■

Those particular rays have already been defined in [18, Equation 2.2].

Notation 4.39. If $\phi : \tilde{K}_+ \longrightarrow \mathbb{Z}^M/\langle e_0\rangle$ is the usual map we set $\mathcal{Z}_M^{\underline{\mathcal{E}}} = \mathcal{X}_\phi^{\underline{\mathcal{E}}}$ (see Definition 3.17) for any sequence $\underline{\mathcal{E}}$ of elements of $K_+^\vee$. Remember that if $\underline{\mathcal{E}}$ is a smooth sequence then $\mathcal{Z}_M^{\underline{\mathcal{E}}}$ is a smooth open subset of $\mathcal{Z}_M$ (see Corollary 3.40) and its points have the description given in Proposition 3.41.

Set Φ_M for the union over all $d > 1$ of the sets of surjective maps $M \longrightarrow \mathbb{Z}/d\mathbb{Z}$. □

Theorem 4.40. Let $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta\in\Phi_M}$. We have

$$\{h \le 1\} = \bigcup_{\eta\in\Phi_M} \mathcal{Z}_M^{\mathcal{E}^\eta}.$$

In particular $\{h \le 1\} \subseteq \mathcal{Z}_M^{\mathrm{sm}}$ and $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories

$$\{(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}} \mid V(z_\eta) \cap V(z_\mu) = \emptyset \text{ if } \eta \neq \mu\} = \pi_{\underline{\mathcal{E}}}^{-1}(\{h \le 1\}) \xrightarrow{\simeq} \{h \le 1\}. \qquad \square$$

Proof. The last part of the statement follows from the first one just applying Theorem 3.44 with $\Theta = \{(\mathcal{E}^\eta)\}_{\eta\in\Phi_M}$. Let k be an algebraically closed field and $B \in \mathrm{D}(M)\text{-Cov}(k)$ with graded basis $\{v_m\}_{m\in M}$ and multiplication ψ.

$\supseteq$. Assume $B \in \mathcal{Z}_M^{\mathcal{E}^\eta}(k)$. If B is a torsor we will have $h_{B/k} = 0$. Otherwise, we can write $\psi = \xi 0^{\mathcal{E}^\eta}$ for some $\xi : K \longrightarrow k^*$. Replacing $\operatorname{Spec} k$ by a geometrical point of the maximal torsor of B/k, we can assume that $M = \mathbb{Z}/d\mathbb{Z}$ and $\eta = \mathrm{id}$. In particular, $H_{B/k} = 0$ and, from the definition of $\mathcal{E}^{\mathrm{id}}$, we get $B \simeq k[x]/(x^d)$. So $h_{B/k} = \dim_k m_B/m_B^2 = 1$.

$\subseteq$. Assume $h_{B/k} = 1$. Set C for the maximal torsor of B/k (see Definition 4.23), $H = H_{B/k}$ and $l = |M/H|$. The equality $h_{B/k} = 1$ means that there exists a unique $\bar{r} \in M/H$ (where $r \in M$) such that $h_{B/k,r} = 1$ and so $C_q[v_r] = B_q \simeq C_q[x]/(x^l)$ for all (maximal) primes q of C. In particular, $B = C[v_r] \simeq C[x]/(x^l)$ and $\bar{r}$ generates M/H. Let $\eta : M \longrightarrow M/H \simeq \mathbb{Z}/l\mathbb{Z}$ be the projection. We want to prove that $B \in \mathcal{Z}_M^{\mathcal{E}^\eta}$. Replacing k by a geometrical point of some fppf extension of k, we can assume $C = k[H]$, that is, $v_h v_{h'} = v_{h+h'}$ if $h, h' \in H$.

Finally, the elements $v_h v_r^i$ for $h \in H$ and $0 \leq i < l$ define an M-graded basis of B/k whose associated multiplication is $0^{\mathcal{E}^\eta}$. ■

Theorem 4.41. Let $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta \in \Phi_M}$ and let X be a locally noetherian and locally factorial scheme. Consider the full subcategories

$$\mathcal{C}_X^1 = \{(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \mid \operatorname{codim}_X V(z_\eta) \cap V(z_\mu) \geq 2 \text{ if } \eta \neq \mu\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

and

$$\mathcal{D}_X^1 = \{Y \xrightarrow{f} X \in \mathrm{D}(M)\text{-Cov}(X) \mid h_f(p) \leq 1 \ \forall p \in X \text{ with } \operatorname{codim}_p X \leq 1\} \subseteq \mathrm{D}(M)\text{-Cov}(X).$$

Then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories

$$\mathcal{D}_X^1 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{C}_X^1) \xrightarrow{\simeq} \mathcal{C}_X^1. \qquad \square$$

Proof. Apply Theorem 3.52 with $\Theta = \{(\mathcal{E}^\eta)\}_{\eta \in \Phi_M}$. ■

Theorem 4.42. Let $\underline{\mathcal{E}} = (\mathcal{E}^\eta)_{\eta \in \Phi_M}$ and let X be a locally noetherian and locally factorial scheme without isolated points and $(\operatorname{char} X, |M|) = 1$, that is, $1/|M| \in \mathcal{O}_X(X)$. Consider the full subcategories

$$\mathrm{Reg}_X^1 = \{Y/X \in \mathrm{D}(M)\text{-Cov}(X) \mid Y \text{ regular in codimension } 1\} \subseteq \mathrm{D}(M)\text{-Cov}(X)$$

and

$$\widetilde{\mathrm{Reg}}_X^1 = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \;\middle|\; \begin{array}{c} \forall\, \mathcal{E} \neq \delta \in \underline{\mathcal{E}} \operatorname{codim}_X V(z_{\mathcal{E}}) \cap V(z_\delta) \geq 2 \\ \forall\, \mathcal{E} \in \underline{\mathcal{E}} \forall p \in X^{(1)} v_p(z_{\mathcal{E}}) \leq 1 \end{array} \right\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X).$$

Then we have an equivalence of categories

$$\widetilde{\mathrm{Reg}}_X^1 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathrm{Reg}_X^1) \xrightarrow{\simeq} \mathrm{Reg}_X^1. \qquad \square$$

Proof. We will make use of Theorem 4.41. If $Y \xrightarrow{f} X \in \mathrm{Reg}_X^1$, $p \in Y^{(1)}$ and $q = f(p)$, then $h_f(q) \leq \dim_{k(p)} m_p/m_p^2 = 1$. So $\mathrm{Reg}_X^1 \subseteq \mathcal{D}_X^1$. So we only have to check that $\widetilde{\mathrm{Reg}}_X^1 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathrm{Reg}_X^1) \subseteq \mathcal{C}_X^1$. Since X is a disjoint union of positive dimensional, integral connected components, we can assume that $X=\operatorname{Spec} R$, where R is a discrete valuation ring. Let $\chi\in\mathcal{C}^1_X$, $A/R\in\mathcal{D}^1_X$ the associated covers, $H=H_{A/R}$ and C be the maximal torsor of A/R. We have to prove that $\chi\in\widetilde{\operatorname{Reg}}^1_X$ if and only if A is regular in codimension 1. Since $D_R(H)$ is etale over R so is also $\operatorname{Spec} C$. It is so easy to check that, replacing R by a localization of C and M with M/H, we can assume that $H=0$. Since $\chi\in\mathcal{C}^1_X$, the multiplication of A over R is of the form $\psi=\mu z^{r\mathcal{E}^\phi}$, where $\mu:K\longrightarrow R^*$ is an M-torsor, z is a parameter of A, $\phi:M\longrightarrow\mathbb{Z}/l\mathbb{Z}$ is an isomorphism and $r=v_R(z_{\mathcal{E}^\phi})$. Moreover, $v_R(z_{\mathcal{E}^\psi})=0$ if $\psi\neq\phi$. Replacing M by $\mathbb{Z}/l\mathbb{Z}$ through ϕ we can assume $\phi=\mathrm{id}$. Finally, since μ induces an (fppf) torsor which is etale over R, replacing R by an etale neighborhood, we can assume $\mu=1$. After these reductions we have $A=R[X]/(X^{|M|}-z^r)$ which is regular in codimension 1 if and only if $r=1$. ■

Remark 4.43. In the above theorem, one can replace the condition "regular in codimension 1" in the definition of Reg^1_X with "normal" thanks to Serre's conditions, since all the fibers involved are Gorenstein. Moreover, note that a locally noetherian and locally factorial scheme X is a disjoint union of integral connected components. Therefore, an isolated point is just a connected component which is $\operatorname{Spec} k$, for a field k. We want to avoid this situation because regularity in codimension 1 for a cover over a field is an empty condition. □

Remark 4.44. Theorem 4.42 is a rewriting of [18, Theorem 2.1 and Corollary 3.1] extended to locally noetherian and locally factorial schemes without isolated points, where an object of $\mathcal{F}_{\underline{\mathcal{E}}}(X)$ is called a building data. □

5 The Locus $h\leq 2$

In this section, we want to give a characterization of the open substack $\{h\leq 2\}\subseteq$ D(M)-Cov as done in Theorem 4.41 for $\{h\leq 1\}$. The general problem we want to solve can be stated as follows.

Problem 5.1. Find a sequence of smooth extremal rays $\underline{\mathcal{E}}$ for M and a collection Θ of smooth sequences with rays in $\underline{\mathcal{E}}$ such that (see Notation 4.39)

$$\{h\leq 2\}=\bigcup_{\underline{\delta}\in\Theta}\mathcal{Z}^{\underline{\delta}}_M$$

or, equivalently, such that, for any algebraically closed field k, the algebras $A \in \mathrm{D}(M)\text{-Cov}(k)$ with $h_{A/k} \leq 2$ are exactly the algebras associated to a multiplication of the form $\psi = \omega 0^{\mathcal{E}}$ where $\omega : K \longrightarrow k^*$ is a group homomorphism and $\mathcal{E} \in \langle \underline{\delta} \rangle_{\mathbb{N}}$ for some $\underline{\delta} \in \Theta$. □

For example in the case $h \leq 1$ the analogous problem is solved taking $\underline{\mathcal{E}} = (\mathcal{E}^{\phi})_{\phi \in \Phi_M}$ and $\Theta = \{(\mathcal{E}) \text{ for } \mathcal{E} \in \underline{\mathcal{E}}\}$ (see Theorem 4.40). Once we have found a pair $\underline{\mathcal{E}}, \Theta$ as in Problem 5.1 we can formally apply Theorems 3.44 and 3.52. This is done in Theorems 5.42 and 5.45.

Similarly to what happens in the case $h \leq 1$, we can restrict our attention to the case when M is generated by two elements m and n and the first problem to solve is to describe M-graded algebras A over a field k generated in these degrees m and n (see Problem 5.9). This is done associating with A an invariant $\bar{q}_A \in \mathbb{N}$ (see Theorem 5.31) and this solution also suggests how to proceed for the next problem, that is, find the sequence $\underline{\mathcal{E}}$ of Problem 5.1.

When M is any finite abelian group, it turns out that the extremal rays $\mathcal{E}$ for M such that $h_{\mathcal{E}} = 2$ correspond to particular sequences of the form $\chi = (r, \alpha, N, \bar{q}, \phi)$, where $r, \alpha, N, \bar{q} \in \mathbb{N}$ and ϕ is a surjective map from M to a group $M_{r,\alpha,N}$ generated by two elements (see Definition 5.6). The sequence of smooth extremal rays "needed" to describe the substack $\{h \leq 2\}$ is composed by the "old" rays $(\mathcal{E}^{\eta})_{\eta \in \Phi_M}$ and by these new rays. Finally, the smooth sequences in the family Θ of Problem 5.1 will all be given by elements of the dual basis of particular $\mathbb{Z}$-basis of K (see Lemma 5.34).

In the last subsection, we will see (Theorem 5.55) that the $\mathrm{D}(M)$-covers of a locally noetherian and locally factorial scheme with no isolated points and with $(\mathrm{char}\, X, |M|) = 1$ whose total space is normal crossing in codimension 1 can be described in the spirit of classification Theorem 4.42 and extending this result.

Notation 5.2. If $m \in M$, we will denote by $o(m)$ the order of m in the group M. □

5.1 Good sequences

In this subsection, we provide some general technical results in order to work with M-graded algebras over local rings. So we will consider given a local ring D, a sequence $\underline{m} = m_1, \dots, m_r \in M$ and $C \in \mathrm{D}(M)\text{-Cov}(D)$ generated in degrees $m_1, \dots, m_r$. Since $\mathrm{Pic}(D) = 0$ for any $u \in M$ we have $C_u \simeq D$. Given $u \in M$, we will call v_u a generator of C_u and we will also use the abbreviation $v_i = v_{m_i}$. Moreover, if $\underline{A} = (A_1, \dots, A_r) \in \mathbb{N}^r$ we will also write

$$v^{\underline{A}} = v_1^{A_1} \cdots v_r^{A_r}.$$

Definition 5.3. A sequence for $u \in M$ is a sequence $\underline{A} \in \mathbb{N}^r$ such that $A_1 m_1 + \cdots + A_r m_r = u$. Such a sequence will be called *good* if the map $C_{m_1}^{A_1} \otimes \cdots \otimes C_{m_r}^{A_r} \longrightarrow C_u$ is surjective, that is, $v^{\underline{A}}$ generates C_u. If $r = 2$, we will talk about pairs instead of sequences. □

Remark 5.4. Any $u \in M$ admits a good sequence since, otherwise, we will have $C_u = (D[v_1, \dots, v_r])_u \subseteq m_D C_u$. If $\underline{A}$ is a good sequence and $\underline{B} \leq \underline{A}$, then also $\underline{B}$ is a good sequence. □

Lemma 5.5. Let $\underline{A}$ and $\underline{B}$ be two sequences for some element of M and assume that $\underline{A}$ is good. Set $\underline{E} = \min(\underline{A}, \underline{B}) = (\min(A_1, B_1), \dots, \min(A_r, B_r))$ and take $\lambda \in D$. Then

$$v^{\underline{B}} = \lambda v^{\underline{A}} \Longrightarrow v^{\underline{B}-\underline{E}} = \lambda v^{\underline{A}-\underline{E}}. \qquad \square$$

Proof. Clearly, we have $v^{\underline{E}}(v^{\underline{B}-\underline{E}} - \lambda v^{\underline{A}-\underline{E}}) = 0$. On the other hand, since $\underline{A} - \underline{E}$ is a good sequence, there exists $\mu \in D$ such that $v^{\underline{B}-\underline{E}} = \mu v^{\underline{A}-\underline{E}}$. Since $\underline{A}$ is a good sequence, substituting we get $v^{\underline{A}}(\mu - \lambda) = 0 \Longrightarrow \mu = \lambda$. ■

5.2 *M*-graded algebras generated in two degrees

Definition 5.6. Given $0 \leq \alpha < N$ and $r > 0$, we set

$$M_{r,\alpha,N} = \mathbb{Z}^2 / \langle (r, -\alpha), (0, N) \rangle. \qquad \square$$

Proposition 5.7. A finite abelian group M with two marked elements $m, n \in M$ generating it is canonically isomorphic to $(M_{r,\alpha,N}, e_1, e_2)$ where $r = \min\{s > 0 \mid sm \in \langle n \rangle\}$, $rm = \alpha n$ and $N = o(n)$. Moreover, we have: $|M| = Nr$, $o(m) = rN/(\alpha, N)$ and

$$m, n \neq 0 \text{ and } m \neq n \Longleftrightarrow N > 1 \quad \text{and} \quad (r > 1 \text{ or } \alpha > 1). \qquad \square$$

Proof. We have

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} r & 0 \\ -\alpha & N \end{smallmatrix}\right)} \mathbb{Z}^2 \longrightarrow M_{r,\alpha,N} \longrightarrow 0 \text{ exact} \Longrightarrow |M_{r,\alpha,N}| = \left| \det \begin{pmatrix} r & 0 \\ -\alpha & N \end{pmatrix} \right| = rN$$

and clearly e_1 and e_2 generate M. Moreover, $M_{r,\alpha,N}/\langle e_2\rangle \simeq \mathbb{Z}/r\mathbb{Z}$ and therefore r is the minimum such that $re_1 \in \langle e_2\rangle$. Finally, it is easy to check that $N = o(e_2)$. If now M, r, α, and N are as in the statement, there exists a unique map $M_{r,\alpha,N} \longrightarrow M$ sending e_1, e_2 to m, n. This map is an isomorphism since it is clearly surjective and $|M| = o(m)o(n)/|\langle m\rangle \cap \langle n\rangle| = o(n)r = |M_{r,\alpha,N}|$. The last equivalence in the statement is now easy to prove. ■

Notation 5.8. In this subsection, we will fix a finite abelian group M generated by two elements $0 \neq m, n \in M$ such that $m \neq n$. Up to isomorphism, this means $M = M_{r,\alpha,N}$ with $m = e_1, n = e_2$ and with the conditions $0 \leq \alpha < N, r > 0, N > 1, (r > 1$ or $\alpha > 1)$.

We will write d_q the only integer $0 < d_q \leq N$ such that $qrm + d_q n = 0$, for $q \in \mathbb{Z}$, or, equivalently, $d_q \equiv -q\alpha \bmod (N)$. □

Problem 5.9. Let k be a field. We want to describe, up to isomorphism, algebras $A \in \mathrm{D}(M)\text{-Cov}(k)$ such that A is generated in degrees m, n and $H_{A/k} = 0$. Thanks to Corollary 4.30, this is equivalent to asking for an algebra A such that $H_{A/k} = 0$ and

$$\{l \in M \mid h_{A/k,l} = 1\} \subseteq \{m, n\}.$$

The solution of this problem is contained in Theorem 5.31. □

In this subsection, we will fix an algebra A as in Problem 5.9, we will consider given a graded basis $\{v_l\}_{l\in M}$ of A and we will denote by ψ the associated multiplication. Note that $H_{A/k} = 0$ means $v_m, v_n \notin A^*$.

Definition 5.10. Define

$$z = \min\{h > 0 \mid \exists i \in \mathbb{N}, \lambda \in k \text{ such that } v_m^h = \lambda v_n^i \text{ and } hm = in\},$$

$$x = \min\{h > 0 \mid \exists i \in \mathbb{N}, \mu \in k \text{ such that } v_n^h = \mu v_m^i \text{ and } hn = im\}.$$

Denote by $0 \leq y < o(n)$ and $0 \leq w < o(m)$ the elements such that $zm = yn$, $xn = wm$, by $\lambda, \mu \in k$ the elements such that $v_m^z = \lambda v_n^y$, $v_n^x = \mu v_m^w$, with the convention that $\lambda = 0$ if $v_n^y = 0$ and $\mu = 0$ if $v_m^w = 0$. Finally, set $\bar{q} = z/r$ and define the map of sets

$$\begin{tikzcd} \{0, 1, \dots, z-1\} \arrow[r, "f"] & \{0, 1, \dots, o(n)\} \\ c \arrow[r, mapsto] & \min\{d \in \mathbb{N} \mid v_m^c v_n^d = 0\} \end{tikzcd}$$

We will also write $\bar{q}_A$, z_A, x_A, y_A, w_A, λ_A, μ_A, and f_A if necessary. □

We will see that A is uniquely determined by $\bar{q}$ and λ up to isomorphism.

Lemma 5.11. Given $l \in M$, there exists a unique good pair (a, b) for l with $0 \leq a < z$. Moreover, $0 \leq b < f(a)$. □

Proof. *Existence.* We know that there exists a good pair (a, b) for l and we can assume that a is minimum. If $a \geq z$ we can write $v_m^a v_n^b = \lambda v_m^{a-z} v_n^{b+y}$. Therefore, $\lambda \neq 0$ and $(a-z, b+y)$ is a good pair for l, contradicting the minimality of a. Finally, $v_m^a v_n^b \neq 0$ means $b < f(a)$.

Uniqueness. Let (a, b) and (a', b') be two good pairs for l and assume $0 \leq a < a' < z$. So there exists $\omega \in k^*$ such that

$$v_m^a v_n^b = \omega v_m^{a'} v_n^{b'} \Longrightarrow v_n^b = \omega v_m^{a'-a} v_n^{b'}.$$

If $b \geq b'$ then $a' - a \geq z$ by definition of z, while if $b < b'$ then v_n is invertible. ■

Definition 5.12. Given $l \in M$, we will write the associated good pair as $(\mathcal{E}_l, \delta_l)$ with $\mathcal{E}_l < z$. We will consider $\mathcal{E}$ and δ as maps $\mathbb{Z}^M / \langle e_0 \rangle \longrightarrow \mathbb{Z}$ and, if necessary, we will also write $\mathcal{E}^A$ and δ^A. □

Notation 5.13. Up to isomorphism, we can change the given basis to

$$v_l = v_m^{\mathcal{E}_l} v_n^{\delta_l}$$

so that the multiplication ψ is given by

$$v_a v_b = v_m^{\mathcal{E}_a + \mathcal{E}_b} v_n^{\delta_a + \delta_b} = \psi_{a,b} v_m^{\mathcal{E}_{a+b}} v_n^{\delta_{a+b}} = \psi_{a,b} v_{a+b}. \tag{5.1}$$

□

Corollary 5.14. f is a decreasing function and

$$f(0) + \cdots + f(z-1) = |M|. \tag{5.2}$$

□

Proof. If (a, b) is a pair such that $0 \leq a < z$ and $0 \leq b < f(a)$ then $v_m^a v_n^b \neq 0$, that is, (a, b) is a good pair for $am + bn$. So

$$\sum_{c=0}^{z-1} f(c) = |\{(a, b) \mid 0 \leq a < z,\ 0 \leq b < f(a)\}| = |M|. \qquad \blacksquare$$

Remark 5.15. The following pairs are good:

$$(z-1)m : (z-1, 0), \quad (x-1)n : (0, x-1), \quad zm = yn : (0, y), \quad xn = wm : (w, 0)$$

that is, $v_m^{z-1}, v_n^{x-1}, v_n^y, v_m^w \neq 0$. In particular, $f(0) \geq x, y+1$ and $f(c) > 0$ for any c. Indeed

$$v_m^{z-1} = \omega v_m^a v_n^b \Longrightarrow v_m^{z-1-a} = \omega v_n^b \Longrightarrow a = z-1,\ b = 0,$$

$$v_m^z = \omega v_m^a v_n^b \Longrightarrow v_m^{z-a} = \omega v_n^b \Longrightarrow a = 0,\ b = y,$$

where (a, b) are good pairs for the given elements and, by symmetry, we get the result. □

Remark 5.16. If $\lambda \neq 0$ or $\mu \neq 0$, then $x = y$, $z = w$ and $\lambda\mu = 1$. Assume for example $\lambda \neq 0$. If $y = 0$ then $v_m^z = \lambda \neq 0$ and so v_m is invertible. So $y > 0$ and, since $v_n^y = \lambda^{-1} v_m^z$, we also have $y \geq x$. Now

$$0 \neq v_m^z = \lambda v_n^y = \lambda\mu v_n^{y-x} v_m^w.$$

So $\mu \neq 0$ and $(y-x, w)$ is a good pair. As before $w \geq z$ and therefore

$$\lambda\mu v_n^{y-x} v_m^{w-z} = 1 \Longrightarrow y = x,\ w = x \quad \text{and} \quad \lambda\mu = 1. \qquad \square$$

Lemma 5.17. Let $a, b \in M$. We have:

- Assume $\mathcal{E}_{a,b} > 0$. If $\delta_{a,b} \leq 0$ then $\mathcal{E}_{a,b} \geq z$, $\delta_{a,b} \geq -y$. Moreover $\psi_{a,b} \neq 0 \iff \lambda \neq 0$, $\mathcal{E}_{a,b} = z$, $\delta_{a,b} = -y (= -x)$ and in this case $\psi_{a,b} = \lambda$.
- Assume $\mathcal{E}_{a,b} < 0$. Then $\mathcal{E}_{a,b} \geq -w$, $\delta_{a,b} \geq x$. Moreover $\psi_{a,b} \neq 0 \iff \mu \neq 0$, $\mathcal{E}_{a,b} = -w (= -z)$, $\delta_{a,b} = x$ and in this case $\psi_{a,b} = \mu$.
- Assume $\mathcal{E}_{a,b} = 0$. Then we have $\delta_{a,b} = 0$ and $\psi_{a,b} = 1$ or $\delta_{a,b} \geq o(n)$ and $\psi_{a,b} = 0$. □

Proof. Set $\psi = \psi_{a,b}$. We start with the case $\mathcal{E}_{a,b} > 0$. From (5.1), we get

$$v_m^{\mathcal{E}_{a,b}} v_n^{\delta_a+\delta_b} = \psi v_n^{\delta_{a+b}}.$$

If $\delta_{a,b} > 0$ then $v_m^{\mathcal{E}_{a,b}} v_n^{\delta_{a,b}} = \psi$ and so $\psi = 0$ since $v_m \notin A^*$. If $\delta_{a,b} \leq 0$ we instead have $v_m^{\mathcal{E}_{a,b}} = \psi v_n^{-\delta_{a,b}}$ and so $\mathcal{E}_{a,b} \geq z$. If $-\delta_{a,b} < y$ then $(0, -\delta_{a,b})$ is good. So we can write

$$v_m^{\mathcal{E}_{a,b}-z} \lambda v_n^{y+\delta_{a,b}} = \psi \Longrightarrow \psi = 0$$

since v_n is not invertible. If $\delta_{a,b} \leq -y$ we have

$$0 \leq \mathcal{E}_{a,b} - z < z, \quad 0 \leq -\delta_{a,b} - y < f(0), \quad (\mathcal{E}_{a,b} - z)m = (-\delta_{a,b} - y)n, \quad v_m^{\mathcal{E}_{a,b}-z} \lambda = \psi v_n^{-\delta_{a,b}-y}$$

and so both $(\mathcal{E}_{a,b} - z, 0)$ and $(0, -\delta_{a,b} - y)$ are good pair for the same element of M. Therefore, we must have $\mathcal{E}_{a,b} = z$, $\delta_{a,b} = -y$ and $\psi = \lambda$.

Now assume $\mathcal{E}_{a,b} = 0$. If $\delta_{a,b} < 0$ then $v_n^{-\delta_{a,b}} \psi = 1$ which is impossible. So $\delta_{a,b} \geq 0$. If $\delta_{a,b} = 0$ clearly $\psi = 1$. If $\delta_{a,b} > 0$ then $v_n^{\delta_{a,b}} = \psi$ and so $\psi = 0$ and $\delta_{a,b} \geq o(n)$.

Finally assume $\mathcal{E}_{a,b} < 0$. From (5.1) we get

$$v_n^{\delta_a+\delta_b} = \psi v_m^{-\mathcal{E}_{a,b}} v_n^{\delta_{a+b}}.$$

We must have $\delta_{a,b} > 0$ since v_m is not invertible. So $v_n^{\delta_{a,b}} = \psi v_m^{-\mathcal{E}_{a,b}}$ and $\delta_{a,b} \geq x$, from which

$$v_n^{\delta_{a,b}-x} \mu v_m^{w} = \psi v_m^{-\mathcal{E}_{a,b}}.$$

Note that, since $0 \leq -\mathcal{E}_{a,b} \leq \mathcal{E}_{a+b} < z$, $(-\mathcal{E}_{a,b}, 0)$ is a good pair. If $w > -\mathcal{E}_{a,b}$ then $\psi = 0$. So assume $w \leq -\mathcal{E}_{a,b}$. Arguing as above, we must have $\delta_{a,b} = x$, $\mathcal{E}_{a,b} = -w$ and $\psi = \mu$. ■

Lemma 5.18. Define

$$A' = k[s,t]/(s^z, s^c t^{f(c)} \text{ for } 0 \leq c < z).$$

Then $A' \in \mathrm{D}(M)\text{-Cov}(k)$ with graduation $\deg s = m$, $\deg t = n$ and it satisfies the requests of Problem 5.9, that is, A' is generated in degrees m, n and $H_{A'/k} = 0$. Moreover, we have

$$\bar{q}_{A'} = \bar{q}_A, \quad z_{A'} = z_A, \quad y_{A'} = y_A, \quad \mathcal{E}^{A'} = \mathcal{E}^A, \quad \delta^{A'} = \delta^A, \quad \lambda_{A'} = \mu_{A'} = 0, \quad f_{A'} = f_A. \qquad \square$$

Proof. Clearly, the elements $s^c t^d$ for $0 \leq c < z$, $0 \leq d < f(c)$ generates A' as a k-space. Since they are $\sum_{c=0}^{z-1} f(c) = |M|$ and they all have different degrees, it is enough to prove that any of them are non-zero. So let (c', d') a pair as always. It is enough to show that $B = k[s, t]/(s^{c'+1}, t^{d'+1}) \longrightarrow A'/(s^{c'+1}, t^{d'+1})$ is an isomorphism. But $c' < z$ implies that $s^z = 0$ in B. If $c' < c$ then $s^c t^{f(c)} = 0$ in B and finally if $c' \geq c$ then $d' + 1 \leq f(c') \leq f(c)$ and so $s^c t^{f(c)} = 0$ in B.

The algebra A' is clearly generated in degrees m, n and $H_{A'/k} = 0$ since $s^z = t^{f(0)} = 0$ and z, $f(0) > 0$. Moreover $s^z = 0 t^y$ implies that $z' = z_{A'} \leq z$. Assume by contradiction $z' < z$. From $0 \neq s^{z'} = \lambda' t^{y'}$ we know that $t^{y'} \neq 0$ so that $y' < f(0)$. Therefore, $(\mathcal{E}_{z'm}, \delta_{z'm}) = (z', 0) = (0, y')$ and so $z' = 0$, which is a contradiction. Then $z' = z$, $y_{A'} = y' = y$. Also $s^z = 0 t^y$ and $t^y \neq 0$ imply $\lambda_{A'} = 0$ and, thanks to Remark 5.16, $\mu_{A'} = 0$. Finally, by construction, we also have $\mathcal{E}^{A'} = \mathcal{E}$, $\delta^{A'} = \delta$ and $f_{A'} = f$. ■

Lemma 5.19. We have

$$d_{\bar{q}} = \max_{1 \leq q \leq \bar{q}} d_q. \qquad \square$$

Proof. Thanks to Lemma 5.18, we can assume $\lambda = 0$ and, therefore, $\mu = 0$. So $v_n^x = 0$, $v_n^{x-1} \neq 0$ and $v_n^y \neq 0$ imply $y < x = f(0)$. Let $1 \leq q < \bar{q}$ and $l = qr$. We have $(\mathcal{E}_l, \delta_l) = (qr, 0)$. If $N - d_q < x = f(0)$ then we will also have $(\mathcal{E}_l, \delta_l) = (0, N - d_q)$ and so $q = 0$, which is not the case. So $N - d_q \geq x > y = N - d_{\bar{q}} \Longrightarrow d_q < d_{\bar{q}}$. ■

Lemma 5.20. Define $\hat{q}$ as the only integers $0 \leq \hat{q} < \bar{q}$ such that

$$d_{\hat{q}} = \min_{0 \leq q < \bar{q}} d_q.$$

If $\lambda = 0$ we have $d_{\hat{q}} \leq x = f(0)$ and

$$f(c) = \begin{cases} x & \text{if } 0 \leq c < \hat{q}r, \\ d_{\hat{q}} & \text{if } \hat{q}r \leq c < z. \end{cases} \qquad \square$$

Proof. We first want to prove that $f(c) = \min(x, d_q \text{ for } 0 \leq qr \leq c)$. Clearly, we have the inequality $\leq$ since $v_n^x = v_m^{qr} v_n^{d_q} = 0$. Set $d = f(c)$ and let (a, b) a good pair for $cm + dn$, so that $v_m^c v_n^d = 0 v_m^a v_n^b$. We cannot have $b \geq d$ since otherwise $v_m^c = 0$ implies $c \geq z$. If $a \geq c$ then $v_n^d = 0$ and so $d = f(c) \geq x$. Conversely, if $a < c$ then $0 \leq c - a = qr \leq c < z$ and $0 < d - b = d_q \leq d = f(c)$.

We are now ready to prove the expression of f. Note that the pairs $(qr, d_q - 1)$, with $0 \leq q < \bar{q}$, are all the possible pairs for $-n$. So there exists a unique $0 \leq \tilde{q} < \bar{q}$ such that $(\tilde{q}r, d_{\tilde{q}} - 1)$ is good. In particular, if $0 \leq q \neq \tilde{q} < \bar{q}$ we have an expression

$$v_m^{qr} v_n^{d_q-1} = 0 v_m^{\tilde{q}} v_n^{d_{\tilde{q}}-1} \Longrightarrow \begin{cases} q < \tilde{q} & \Longrightarrow \quad v_n^{d_q-1} = 0 \quad \Longrightarrow \quad d_q \geq x, \\ q > \tilde{q} & \Longrightarrow \quad d_q > d_{\tilde{q}}. \end{cases}$$

Since $v_n^{d_{\tilde{q}}-1} \neq 0$ we must have $d_{\hat{q}} \leq x$. This shows that $\tilde{q} = \hat{q}$ and the expression of f. Finally, if $\bar{q} > 1$ then $\hat{q} > 0$ and so $d_{\hat{q}} \leq x = f(0)$ since f is a decreasing function. If $\bar{q} = 1$ then $\hat{q} = 0$ and so $N = d_{\hat{q}} = f(0) \leq x \leq N$. ■

Definition 5.21. We will continue to use notation from Lemma 5.20 for $\hat{q}$ and we will also write $\hat{q}_A$ if necessary. ☐

5.3 The invariant $\bar{q}$

Lemma 5.22. Let $\beta, N \in \mathbb{N}$, with $N > 1$, and define $d_q^\beta = d_q$, for $q \in \mathbb{Z}$, the only integer $0 < d_q \leq N$ such that $d_q \equiv q\beta \bmod N$. Set

$$\Omega_{\beta,N} = \{0 < q \leq o(\beta, \mathbb{Z}/N\mathbb{Z}) = N/(N, \beta) \mid d_{q'} < d_q \text{ for any } 0 < q' < q\},$$

set q_n for the nth element of it and denote by $0 \leq \hat{q} < q_n$ the only number such that

$$d_{\hat{q}} = \min_{0 \leq q < q_n} d_q.$$

Then we have relations $\hat{q}N + q_n d_{\hat{q}} - \hat{q} d_{q_n} = N$ and, if $n > 1$, $q_n = q_{n-1} + \hat{q}$, $d_{q_n} = d_{q_{n-1}} + d_{\hat{q}}$ and $d_{q_{n-1}} + d_q > N$ for $q < \hat{q}$. ☐

Proof. First of all note that all is defined also in the extremal case $\beta = 0$. In this case, $\Omega_{\beta,N} = \{1\}$. Assume first $n > 1$. Set $\tilde{q} = q_n - q_{n-1}$ so that $d_{q_n} = d_{q_{n-1}} + d_{\tilde{q}}$ since $d_{q_n} > d_{q_{n-1}}$. Assume by contradiction that $\tilde{q} \neq \hat{q}$. Since $\tilde{q} < q_n$ we have $d_{\hat{q}} < d_{\tilde{q}}$. Let also $q' = q_n - \hat{q}$ and, as above, we can write $d_{q_n} = d_{q'} + d_{\hat{q}}$. Now

$$d_{q_n} - d_{q'} = d_{\hat{q}} < d_{\tilde{q}} = d_{q_n} - d_{q_{n-1}} \Longrightarrow d_{q_{n-1}} < d_{q'}.$$

Since $q_{n-1} \in \Omega_{\beta,N}$, we must have $q' > q_{n-1}$, which is a contradiction because otherwise, being $q' < q_n$, we must have $q' = q_n$. So $\tilde{q} = \hat{q}$. For the last relation note that, since q_n is the first $q > q_{n-1}$ such that $d_q > d_{q_{n-1}}$, then $\hat{q}$ is the first such that $d_{q_{n-1}} + d_{\hat{q}} \leq N$.

Now consider the first relation. We need to do induction on all the β. So we will write d_q^β and q_n^β in order to remember that those numbers depend on to β. The induction statement on $1 \leq q < N$ is: for any $0 \leq \beta < N$ and for any n such that $q_n^\beta \leq q$ the required formula holds. The base step is $q = 1$. In this case, we have $n = 1$, $q_1 = 1$, $\hat{q} = 0$, $d_0 = N$ and the formula can be proved directly. For the induction step, we can assume $q > 1$ and $n > 1$. We will write $\hat{q}_n^\beta$ for the $\hat{q}$ associated to n and β. First of all note that, by the relations proved above, we can write

$$\hat{q}_n^\beta N + q_n^\beta d^\beta_{\hat{q}_n^\beta} - \hat{q}_n^\beta d^\beta_{q_n^\beta} = \hat{q}_n^\beta N + q_{n-1}^\beta d^\beta_{\hat{q}_n^\beta} - \hat{q}_n^\beta d^\beta_{q_{n-1}^\beta}$$

and so we have to prove that the second member equals N. If $\hat{q}_n^\beta \leq q_{n-1}^\beta$ then $\hat{q}_{n-1}^\beta = \hat{q}_n^\beta$ and the formula is true by induction on $q - 1 \geq q_{n-1}^\beta$. So assume $\hat{q}_n^\beta > q_{n-1}^\beta$ and set $\alpha = N - \beta$. Clearly, we will have

$$o = o(\alpha, \mathbb{Z}/N\mathbb{Z}) = o(\beta, \mathbb{Z}/N\mathbb{Z}) \quad \text{and} \quad d_q^\beta + d_q^\alpha = N \quad \text{for any } 0 < q < o.$$

Moreover,

$$d^\beta_{\hat{q}_n^\beta} < d_q^\beta \text{ for any } 0 < q < q_n^\beta \Longrightarrow d^\alpha_{\hat{q}_n^\beta} > d_q^\alpha \text{ for any } 0 < q < \hat{q}_n^\beta \Longrightarrow \exists l \text{ s.t. } q_l^\alpha = \hat{q}_n^\beta$$

and

$$d^\beta_{q_{n-1}^\beta} \geq d_q^\beta \text{ for any } 0 < q < q_n^\beta \Longrightarrow d^\alpha_{q_{n-1}^\beta} \leq d_q^\alpha \text{ for any } 0 \leq q < q_l^\alpha = \hat{q}_n^\beta \Longrightarrow \hat{q}_l^\alpha = q_{n-1}^\beta.$$

Using induction on $q_l^\alpha = \hat{q}_n^\beta < q_n^\beta \leq q$, we can finally write

$$\begin{aligned} N &= \hat{q}_l^\alpha N + q_l^\alpha d^\alpha_{\hat{q}_l^\alpha} - \hat{q}_l^\alpha d^\alpha_{q_l^\alpha} = q_{n-1}^\beta N + \hat{q}_n^\beta d^\alpha_{q_{n-1}^\beta} - q_{n-1}^\beta d^\alpha_{\hat{q}_n^\beta} \\ &= q_{n-1}^\beta N + \hat{q}_n^\beta (N - d^\beta_{q_{n-1}^\beta}) - q_{n-1}^\beta (N - d^\beta_{\hat{q}_n^\beta}) = \hat{q}_n^\beta N + q_{n-1}^\beta d^\beta_{\hat{q}_n^\beta} - \hat{q}_n^\beta d^\beta_{q_{n-1}^\beta}. \end{aligned}$$

■

We continue to keep notation from Notation 5.8. With d_q we will always mean $d_q^{N-\alpha}$ as in Lemma 5.22. Lemma 5.19 can be restated as follows:

Proposition 5.23. Let A be an algebra as in Problem 5.9. Then $\bar{q}_A \in \Omega_{N-\alpha,N}$. ☐

So given an algebra A as in Problem 5.9, we can associate to it the number $\bar{q}_A \in \Omega_{N-\alpha,N}$. Conversely, we will see that any $\bar{q} \in \Omega_{N-\alpha,N}$ admits an algebra A as in Problem 5.9 such that $\bar{q} = \bar{q}_A$. It turns out that all the objects z_A, y_A, f_A, $\mathcal{E}^A$, δ^A, $\hat{q}_A$ and, if $\lambda_A = 0$, x_A, w_A associated to A only depend on $\bar{q}_A$. Therefore, in this subsection, given $\bar{q} \in \Omega_{N-\alpha,N}$, we will see how to define such objects independently from an algebra A.

In this subsection, we will consider given an element $\bar{q} \in \Omega_{N-\alpha,N}$.

Definition 5.24. Set $\hat{q}$ for the only integer $0 \le \hat{q} < \bar{q}$ such that $d_{\hat{q}} = \min_{0 \le q < \bar{q}} d_q$, $q' = \bar{q} - \hat{q}$, $z = \bar{q}r$, $y = N - d_{\bar{q}}$,

$$x = \begin{cases} N - d_{q'} & \text{if} \quad \bar{q} > 1, \\ N & \text{if} \quad \bar{q} = 1, \end{cases} \qquad w = \begin{cases} q'r & \text{if} \quad \bar{q} > 1, \\ 0 & \text{if} \quad \bar{q} = 1, \end{cases} \qquad f(c) = \begin{cases} x & \text{if} \quad 0 \le c < \hat{q}r, \\ d_{\hat{q}} & \text{if} \quad \hat{q}r \le c < z. \end{cases}$$

We will also write $\hat{q}_{\bar{q}}$, $q'_{\bar{q}}$, $z_{\bar{q}}$, $x_{\bar{q}}$, $f_{\bar{q}}$, $y_{\bar{q}}$, and $w_{\bar{q}}$ if necessary. ☐

Remark 5.25. Using notation from Lemma 5.22, we have $\bar{q} = q_n$ for some n and, if $n > 1$, that is, $\bar{q} > 1$, $q_{n-1} = q'$. Note that $zm = yn$, $wm = xn$, $y < x$, $w < z$. Moreover, from Lemma 5.22 and from a direct computation if $\bar{q} = 1$, we obtain $zx - yw = |M|$. Finally, if $\bar{q} > 1$ one has relations $\hat{q}r = z - w$ and $d_{\hat{q}} = x - y$. ☐

Lemma 5.26. We have that:

(1) f is a decreasing function and $\sum_{c=0}^{z-1} f(c) = |M|$;

(2) any element $t \in M$ can be uniquely written as

$$t = Am + Bn \quad \text{with} \quad 0 \le A < z, 0 \le B < f(A).$$ ☐

Proof. (1) If $\bar{q} = 1$ it is enough to note that $\hat{q} = 0$, $d_0 = N$ and $Nr = |M|$. So assume $\bar{q} > 1$. We have $x = N - d_{q'} \ge d_{\hat{q}}$ since $d_{\bar{q}} = d_{q'} + d_{\hat{q}}$ and

$$\sum_{c=0}^{z-1} f(c) = \hat{q}rx + (\bar{q}r - \hat{q}r)d_{\hat{q}} = (z - w)x + w(x - y) = zx - wy = |M|.$$

(2) First of all note that the expressions of the form $Am+Bn$ with $0 \leq A < z$, $0 \leq B < f(A)$ are $\sum_{c=0}^{z-1} f(c) = |M|$. So it is enough to prove that they are all distinct. Assume that we have expressions $Am+Bn = A'm+B'n$ with $0 \leq A' \leq A < z$, $0 \leq B < f(A), 0 \leq B' < f(A')$.

$A' = B' = 0$, that is, $Am+Bn=0$. If $A=0$ then $B=0$ since $f(0)=x \leq N$. If $A>0$, we can write $A=qr$ for some $0<q<\bar{q}$. In particular, $\bar{q}>1$ and $B=d_q<f(A)$. If $q<\hat{q}$ then $f(A)=x=N-d_{q'}>d_q$ contradicting Lemma 5.22, while if $q \geq \hat{q}$ then $f(A)=d_{\hat{q}} \leq d_q$.

$A' = B = 0$, that is, $Am = B'n$. If $A=0$ then $B'=0$ as above. If $A>0$ we can write $A=qr$ for some $0<q<\bar{q}$. Again $\bar{q}>1$. In particular, $B'=N-d_q<f(0)=x=N-d_{q'}$ and so $d_{q'}<d_q$, while $d_{q'}=\max_{0<q<\bar{q}} d_q$.

General case. We can write $(A-A')m+Bn=B'n$ and we can reduce the problem to the previous cases since if $B \geq B'$ then $B-B' \leq B < f(A) \leq f(A-A')$, while if $B<B'$ then $B'-B \leq B' < f(A') \leq f(0)$. ■

Definition 5.27. Given $l \in M$ we set $(\mathcal{E}_l, \delta_l)$ the unique pair for l such that $0 \leq \mathcal{E}_t < z, 0 \leq \delta_t < f(\mathcal{E}_t)$ and we will consider $\mathcal{E}, \delta$ as maps $\mathbb{Z}^M/\langle e_0 \rangle \longrightarrow \mathbb{Z}$. We will also write $\mathcal{E}^{\bar{q}}$ and $\delta^{\bar{q}}$ if necessary. ☐

Proposition 5.28. Let A be an algebra as in Problem 5.9. Then

$$z_A = z_{\bar{q}_A}, \quad y_A = y_{\bar{q}_A}, \quad \hat{q}_A = \hat{q}_{\bar{q}_A}, \quad \mathcal{E}^A = \mathcal{E}^{\bar{q}_A}, \quad \delta^A = \delta^{\bar{q}_A}, \quad f_A = f_{\bar{q}_A}$$

and, if $\lambda_A = 0$, then $x_A = x_{\bar{q}_A}, w_A = w_{\bar{q}_A}$. ☐

Proof. Set $\bar{q} = \bar{q}_A$. Then $z_A = \bar{q}r = z_{\bar{q}}$ and $z_A m = y_A n = y_{\bar{q}} n$ implies $y_A = y_{\bar{q}}$. Also $\hat{q}_A = \hat{q}_{\bar{q}}$ by definition. Taking into account Lemma 5.18 we can now assume $\lambda_A = 0$. We claim that all the remaining equalities follow from $x_A = x_{\bar{q}}$. Indeed, clearly $w_A = w_{\bar{q}}$. Also by definition of $f_{\bar{q}}$ and thanks to Lemma 5.20 we will have $f_A = f_{\bar{q}}$ and therefore $\mathcal{E}^A = \mathcal{E}^{\bar{q}}, \delta^A = \delta^{\bar{q}}$, that conclude the proof.

We now show that $x_A = x_{\bar{q}}$. If $\bar{q}=1$ then $\hat{q}=0$ and so, from Lemma 5.20, we have $d_{\hat{q}} = N = x_A = x_1$. If $\bar{q}>1$, by definition of $f_{\bar{q}}$ and thanks to Lemmas 5.26 and 5.20, we can write

$$|M| = \sum_{c=0}^{z_{\bar{q}}-1} f_{\bar{q}}(c) = r\hat{q}_{\bar{q}} x_{\bar{q}} + (z_{\bar{q}} - \hat{q}_{\bar{q}} r) d_{\hat{q}_{\bar{q}}} = \sum_{c=0}^{z_A-1} f_A(c) = r\hat{q}_A x_A + (z_A - \hat{q}_A r) d_{\hat{q}_A}$$

and so $x_A = x_{\bar{q}}$. ■

Definition 5.29. Define the M-graded $\mathbb{Z}[a,b]$-algebra

$$A^{\bar{q}} = \mathbb{Z}[a,b][s,t]/(s^z - at^y, t^x - bs^w, s^{\hat{q}r}t^{d_{\hat{q}}} - a^\gamma b) \quad \text{where} \quad \gamma = \begin{cases} 0 & \text{if} \quad \bar{q} = 1, \\ 1 & \text{if} \quad \bar{q} > 1 \end{cases}$$

with M-graduation $\deg s = m$, $\deg t = n$. Given a ring homomorphism $\mathbb{Z}[a,b] \longrightarrow C$, that is elements $a_0, b_0 \in C$, we will also write $A^{\bar{q}}_{a_0,b_0} = A^{\bar{q}} \otimes_{\mathbb{Z}[a,b]} C$. ☐

Proposition 5.30. $A^{\bar{q}} \in \mathrm{D}(M)\text{-}\mathrm{Cov}(\mathbb{Z}[a,b])$, it is generated in degrees m, n and $\{v_l = s^{\mathcal{E}_l}t^{\delta_l}\}_{l \in M}$ is an M-graded basis for it. ☐

Proof. We have to prove that, for any $l \in M$, $(A^{\bar{q}})_l = \mathbb{Z}[a,b]v_l$ and we can check this over a field k, that is, considering $A = A^{\bar{q}}_{a,b}$ with $a, b \in k$. We first consider the case $a, b \in k^*$, so that $s, t \in A^*$. Let $\pi : \mathbb{Z}^2 \longrightarrow M$ the map such that $\pi(e_1) = m$, $\pi(e_2) = n$. The set $T = \{(a,b) \in \operatorname{Ker} \pi \mid s^a t^b \in k^*\}$ is a subgroup of $\operatorname{Ker} \pi$ such that $(z, -y), (-w, x) \in T$. Since $\det \left(\begin{smallmatrix} z & -w \\ -y & x \end{smallmatrix} \right) = zx - wy = |M|$ we can conclude that $T = \operatorname{Ker} \pi$. Therefore, v_l generate $(A^{\bar{q}})_l$ since for any $c, d \in \mathbb{N}$ we have $s^c t^d / v_{cm+dn} \in k^*$ and $0 \neq v_l \in A^*$.

Now assume that $a = 0$. If $\bar{q} = 1$ then $\hat{q} = w = 0$, $d_{\hat{q}} = x = N$ and so $A = k[s,t]/(s^z, t^N - b)$ satisfies the requests. If $\bar{q} > 1$ it is easy to see that v_l generates A_l. On the other hand, $\dim_k A = |\{(A,B) \mid 0 \leq A < z, 0 \leq B < x, A \leq \hat{q}r \text{ or } B \leq d_{\hat{q}}\}| = zx - (z - \hat{q}r)(x - d_{\hat{q}}) = zx - yw = |M|$. The case $b = 0$ is similar. ■

Theorem 5.31. Let k be a field. If $\bar{q} \in \Omega_{N-\alpha,N}$ and $\lambda \in k$, with $\lambda = 0$ if $\bar{q} = N/(\alpha, N)$, then

$$A_{\bar{q},\lambda} = k[s,t]/(s^{z_{\bar{q}}} - \lambda t^{y_{\bar{q}}}, t^{x_{\bar{q}}}, s^{\hat{q}_{\bar{q}}r}t^{d_{\hat{q}_{\bar{q}}}})$$

is an algebra as in Problem 5.9 with $\bar{q}_{A_{\bar{q},\lambda}} = \bar{q}$ and $\lambda_{A_{\bar{q},\lambda}} = \lambda$. Conversely, if A is an algebra as in Problem 5.9 then $\bar{q}_A \in \Omega_{N-\alpha,N}$, $\lambda_A \in k$, $\lambda_A = 0$ if $\bar{q}_A = N/(\alpha, N)$ and $A \simeq A_{\bar{q}_A,\lambda_A}$. ☐

Proof. Consider $A = A_{\bar{q},\lambda}$, which is just $A^{\bar{q}}_{\lambda,0}$. Clearly, $t \notin A^*$. On the other hand, $s \notin A^*$ since $y = 0 \iff z = o(m) \iff \bar{q} = N/(\alpha, N)$. Therefore, $H_{A/k} = 0$ and A is an algebra as in Problem 5.9. Moreover, clearly $\bar{q}_A \leq \bar{q}$. If by contradiction this inequality is strict, we will have a relation $s^{qr} = \omega t^{y'}$ with $0 \leq q < \bar{q}$. Since $s^{qr} = v_{qrm} \neq 0$ we will have that $t^{y'} \neq 0$ and $y' < x$, a contradiction thanks to Lemma 5.26. In particular, $\lambda = \lambda_A$.

Now let A be as in Problem 5.9 and set $\bar{q} = \bar{q}_A$, $\lambda = \lambda_A$. We already know that $\bar{q} \in \Omega_{N-\alpha,N}$ (see Proposition 5.23). We claim that the map $A_{\bar{q},\lambda} \longrightarrow A$ sending s, t to v_m, v_n is well defined and so an isomorphism. Indeed, we have $v_m^z = \lambda v_n^y$ by definition and, thanks to Proposition 5.28, we have $v_m^{\hat{q}r} v_n^{d_{\hat{q}}} = 0$ since $d_{\hat{q}} = f_A(\hat{q}r)$ and $v_n^x = 0$ since $f_A(0) = x$. Finally, if $\bar{q} = N/(\alpha, N)$ then $y = y_A = 0$ and $z = o(m)$, so that $\lambda_A = v_m^{o(m)} = 0$. ■

Corollary 5.32. If k is an algebraically closed field then, up to graded isomorphism, the algebras as in Problem 5.9 are exactly $A_{\bar{q},1}$ if $\bar{q} \in \Omega_{N-\alpha,N} - \{N/(\alpha, N)\}$ and $A_{\bar{q},0}$ if $\bar{q} \in \Omega_{N-\alpha,N}$. □

Proof. Clearly the above algebras cannot be isomorphic. Conversely, if $\lambda \in k^*$ (and $\bar{q} < N/(\alpha, N)$) the transformation $t \longrightarrow \sqrt[y]{\lambda} t$ with $y = y_{\bar{q}}$ yields an isomorphism $A_{\bar{q},\lambda} \simeq A_{\bar{q},1}$. ■

5.4 Smooth extremal rays for $h \leq 2$

In this subsection, we continue to keep notation from Notation 5.8, that is, $M = M_{r,\alpha,N}$ and we will consider given an element $\bar{q} \in \Omega_{N-\alpha,N}$.

Remark 5.33. We have $z = 1 \iff \bar{q} = r = 1$ and $x = 1 \iff \bar{q} = N$. Indeed the first relation is clear, while for the second one note that, by definition of x and since $N > 1$, we have $x = 1 \iff d_{q'} = N - 1 \iff \bar{q} = N/(\alpha, N), (\alpha, N) = 1$. □

Lemma 5.34. The vectors of K_+

$$\begin{aligned} &v_{cm,dn}, \quad 0 < c < z, \quad 0 < d < f(c), \\ &v_{m,im}, \quad 0 < i < z - 1, \\ &v_{n,jn}, \quad 0 < j < x - 1, \\ &v_{m,(z-1)m} \quad \text{if} \quad z > 1, \\ &v_{n,(x-1)n} \quad \text{if} \quad x > 1, \end{aligned} \tag{5.3}$$

form a basis of K. Assume $\bar{q}r \neq 1$ and $\bar{q} \neq N$, that is, $z, x > 1$, and denote by Λ and Δ the last two terms of the dual basis of (5.3). Then $\Lambda, \Delta \in K_+^\vee$ and they form a smooth sequence. Moreover, $\Lambda = 1/|M|(x\mathcal{E} + w\delta)$, $\Delta = 1/|M|(y\mathcal{E} + z\delta)$ and

$$\Lambda_{m,-m} = \Delta_{n,-n} = 1, \quad \Lambda_{n,-n} = \begin{cases} 0 & \text{if} \quad \bar{q} = 1, \\ 1 & \text{otherwise}, \end{cases} \quad \Delta_{m,-m} = \begin{cases} 0 & \text{if} \quad \bar{q} = N/(\alpha, N), \\ 1 & \text{otherwise}. \end{cases}$$ □

Proof. Note that we cannot have $z = x = 1$ since otherwise $|M| = f(0) = x = 1$, that is, $M = 0$. The vectors of (5.3) are at most $\operatorname{rk} K$ since

$$\sum_{c=1}^{z-1}(f(c)-1)+z-2+x-2+2=\sum_{c=0}^{z-1}(f(c)-1)+z-1=|M|-z+z-1=|M|-1=\operatorname{rk} K.$$

If $z = 1$ then (5.3) is $v_{n,n}, \dots, v_{n,(x-1)n}$. So $x = |M| = N$, that is, n generates M, and (5.3) is a base of K. In the same way, if $x = 1$, then m generates M and (5.3) is a base of K.

So we can assume that $z, x > 1$. The functions $\mathcal{E}$ and δ define a map $\mathbb{Z}^M/\langle e_0\rangle \xrightarrow{(\mathcal{E},\delta)} \mathbb{Z}^2$. Denote by K' the subgroup of K generated by the vectors in (5.3), except the last two lines. We claim that $(\mathcal{E}, \delta)_{|K'} = 0$. This follows by a direct computation just observing that if we have an expression $Am + Bn$ as in Lemma 5.26, (2) then $(\mathcal{E}, \delta)(e_{Am+Bn}) = (A, B)$. Consider the diagram

$$\begin{tikzcd}
\sigma(e_1)=v_{m,(z-1)m}, & \sigma(e_2)=v_{n,(x-1)n} & & & & \\
\mathbb{Z}^2 \arrow[r, "\sigma"] \arrow[rrr, bend right, "U"'] & K/K' \arrow[r, hook] & \mathbb{Z}^M/\langle e_0, K'\rangle \arrow[r, "(\mathcal{E},\delta)"] & \mathbb{Z}^2 \arrow[r, "\tau"'] \arrow[rr, bend left, "\pi"] & \mathbb{Z}^M/\langle e_0, K'\rangle \arrow[r, "p"'] & M \\
& & & \tau(e_1)=e_m, & \tau(e_2)=e_n & p(e_l)=l
\end{tikzcd}$$

We have $(\mathcal{E}, \delta)(v_{m,(z-1)m}) = (z, -y)$ since $y < x = f(0)$ and $(\mathcal{E}, \delta)(v_{n,(x-1)n}) = (-w, x)$ since $w < z$. So $|\det U| = zx - yw = |M|$ and, since $\pi \circ U = 0$, U is an isomorphism onto $\operatorname{Ker} \pi$. Moreover, $\tau^{-1} = (\mathcal{E}, \delta)$ since $e_l \equiv \mathcal{E}_l e_m + \delta_l e_n \bmod K'$. It follows that σ is an isomorphism and so (5.3) is a basis of K.

Consider now the second part of the statement. Clearly, $\Lambda, \Delta \in \langle \mathcal{E}, \delta\rangle_{\mathbb{Q}}$. Therefore, we have

$$\Lambda = a\mathcal{E} + b\delta, \quad \begin{cases} \Lambda(v_{m,(z-1)m}) = 1 = az - yb \\ \Lambda(v_{n,(x-1)n}) = 0 = xb - aw \end{cases} \implies \begin{cases} a = x/|M|, \\ b = w/|M| \end{cases}$$

and the analogous relation for Δ follows in the same way. Now note that, thanks to Theorem 5.31 and Proposition 5.28, we have that $\mathcal{E} = \mathcal{E}^A, \delta = \delta^A$ for an algebra A as in Problem 5.9 with $\bar{q}_A = \bar{q}$, $\lambda_A = 0$ and sharing the same invariants of $\bar{q}$. So we can apply Lemma 5.17. We want to prove that $\Lambda, \Delta \in K_+^\vee$ so that they form a smooth sequence by construction. Assume first that $\mathcal{E}_{a,b} > 0$. Clearly, $\Lambda_{a,b}, \Delta_{a,b} \geq 0$ if $\delta_{a,b} \geq 0$. On the other hand, if $\delta_{a,b} < 0$ we know that $\mathcal{E}_{a,b} \geq z$ and $\delta_{a,b} \geq -y$ and so

$$|M|\Lambda_{a,b} = x\mathcal{E}_{a,b} + w\delta_{a,b} \geq xz - yw = |M| \quad \text{and} \quad |M|\Delta_{a,b} = y\mathcal{E}_{a,b} + z\delta_{a,b} \geq yz - zy = 0.$$

The other cases follows in the same way. It remains to prove the last relations. Since $-n = \hat{q}rm + (d_{\hat{q}} - 1)n$, we have $\mathcal{E}_{n,-n} = \hat{q}r$ and $\delta_{n,-n} = d_{\hat{q}}$. Using the relation $zx - wy = |M|$ the values of $\Lambda_{n,-n}$, $\Delta_{n,-n}$ can be checked by a direct computation. Similarly, considering the relations $-m = (\hat{q}r - 1)m + d_{\hat{q}}n$ if $1 < \bar{q}$, $-m = (r-1)m + (N-\alpha)n$ if $\bar{q} = 1$ and $\alpha \neq 0$, $-m = (r-1)m$ if $\alpha = 0$, we can compute the values of $\Lambda_{m,-m}$ and $\Delta_{m,-m}$. ■

Proposition 5.35. The multiplication of $A^{\bar{q}}$ (see Definition 5.29) with respect to the basis $v_l = v_m^{\mathcal{E}_l} v_n^{\delta_l}$ is: $a^{\mathcal{E}^{\phi}}$ if $\bar{q} = N$, where $\phi : M \xrightarrow{\simeq} \mathbb{Z}/|M|\mathbb{Z}$, $\phi(m) = 1$; $b^{\mathcal{E}^{\eta}}$ if $\bar{q}r = 1$, where $\eta : M \xrightarrow{\simeq} \mathbb{Z}/|M|\mathbb{Z}$, $\phi(n) = 1$; $a^{\Lambda}b^{\Delta}$ if $\bar{q}r \neq 1, \bar{q} \neq N$, where Λ and Δ are the rays defined in Lemma 5.34. □

Proof. In the proof of Proposition 5.30, we have seen that if $x = 1$ $(\bar{q} = N)$, then $M = \langle m \rangle$ and $A^{\bar{q}} = \mathbb{Z}[a,b][s]/(s^{|M|} - a)$, while if $z = 1$ $(\bar{q}r = 1)$ then $M = \langle n \rangle$ and $A^{\bar{q}} = \mathbb{Z}[a,b][t]/(t^{|M|} - b)$. So we can assume $x, z > 1$. Let B the D(M)-cover over $\mathbb{Z}[a,b]$ given by multiplication $\psi = a^{\Lambda}b^{\Delta}$ and denote by $\{\omega_l\}_{l \in M}$ a graded basis (inducing ψ). By definition of Λ and Δ, we have $\omega_l = \omega_m^{\mathcal{E}_l}\omega_n^{\delta_l}$ for any $l \in M$ and $\psi_{m,(z-1)m} = a$, $\psi_{n,(x-1)n} = b$. Therefore,

$$\omega_m^z = \omega_m\omega_{(z-1)m} = a\omega_{zm} = a\omega_{yn} = a\omega_n^y,\ \omega_n^x = \omega_n\omega_{(x-1)n} = b\omega_{xn} = b\omega_{wm} = b\omega_m^w$$

and, checking both cases $\bar{q} = 1$ and $\bar{q} > 1$, $\omega_m^{\hat{q}r}\omega_n^{d_{\hat{q}}} = \omega_{-n}\omega_n = a^{\Lambda_{n,n}}b^{\Delta_{n,n}} = a^{\gamma}b$. In particular, we have an isomorphism $A^{\bar{q}} \longrightarrow B$ sending v_m, v_n to ω_m, ω_n. ■

Notation 5.36. From now on M will be any finite abelian group. If $\phi : M \longrightarrow M_{r,\alpha,N}$ is a surjective map, r, α, N satisfy the conditions of Notation 5.8, $\bar{q} \in \Omega_{N-\alpha,N}$ with $\bar{q}r \neq 1, \bar{q} \neq N$ then we set $\Lambda^{r,\alpha,N,\bar{q},\phi} = \Lambda \circ \phi_*$, $\Delta^{r,\alpha,N,\bar{q},\phi} = \Delta \circ \phi_*$, where Λ and Δ are the rays defined in Lemma 5.34 with respect to $r, \alpha, N, \bar{q}$. If $\phi = \mathrm{id}$ we will omit it. □

Definition 5.37. Set

$$\Sigma_M = \left\{ (r,\alpha,N,\bar{q},\phi) \;\middle|\; \begin{array}{c} 0 \leq \alpha < N, r > 0, N > 1,\ (r > 1 \text{ or } \alpha > 1) \\ \bar{q} \in \Omega_{N-\alpha,N}, \bar{q}r \neq 1, \bar{q}\alpha \not\equiv 1 \bmod N \\ \bar{q} \neq N/(\alpha,N), \phi : M \longrightarrow M_{r,\alpha,N} \text{ surjective} \end{array} \right\}$$

and $\Delta^* : \Sigma_M \longrightarrow \{\text{smooth extremal rays of } M\}$. □

Remark 5.38. Since e_2 and e_1 generate $M_{r,\alpha,N}$, there exist unique $r^\vee, \alpha^\vee$, and $N^\vee$ with an isomorphism $(-)^\vee : M_{r,\alpha,N} \longrightarrow M_{r^\vee,\alpha^\vee,N^\vee}$ sending e_2, e_1 to e_1, e_2. One can check that $r^\vee = (\alpha, N)$, $N^\vee = rN/(\alpha, N)$ and $\alpha^\vee = \tilde{q}r$, where $\tilde{q}$ is the only integer $0 \leq \tilde{q} < N/(\alpha, N)$ such that $\tilde{q}\alpha \equiv (\alpha, N) \bmod N$.

If A is an algebra as in Problem 5.9 for $M_{r,\alpha,N}$, then, through $(-)^\vee$, A can be thought of as a $M_{r^\vee,\alpha^\vee,N^\vee}$-cover, that we will denote by $A^\vee$, and $A^\vee$ is an algebra as in Problem 5.9 with respect to $M_{r^\vee,\alpha^\vee,N^\vee}$, with $\bar{q}_{A^\vee} = x_A/(\alpha, N)$, $\lambda_{A^\vee} = \mu_A$. We can define a bijection $(-)^\vee : \Omega_{N-\alpha,N} - \{N/(N,\alpha)\} \longrightarrow \Omega_{N^\vee-\alpha^\vee,N^\vee} - \{N^\vee/(\alpha^\vee, N^\vee)\}$ in the following way. Given $\bar{q}$ take an algebra A as in Problem 5.9 for $M_{r,\alpha,N}$ with $\bar{q}_A = \bar{q}$ and $\lambda_A \neq 0$, which exists thanks to Theorem 5.31, and set $\bar{q}^\vee = \bar{q}_{A^\vee}$. Taking into account Remark 5.16 and Proposition 5.28, $\bar{q}^\vee = y_{\bar{q}}/(\alpha, N)$ since $x_A = y_A = y_{\bar{q}}$ and $(-)^\vee$ is well defined and bijective since $\lambda_{A^\vee} = \mu_A = \lambda_A^{-1}$. Note that the condition $\bar{q}\alpha \equiv 1 \bmod N$ is equivalent to $r^\vee = 1$ and $\bar{q}^\vee = 1$.

Finally, if $\phi : M \longrightarrow M_{r,\alpha,N}$ is a surjective morphism then we set $\phi^\vee = (-)^\vee \circ \phi : M \longrightarrow M_{r^\vee,\alpha^\vee,N^\vee}$. Note that in any case we have the relation $(-)^{\vee^\vee} = \mathrm{id}$. In particular, since $1^\vee = \alpha/r^\vee$, $\bar{q} = \alpha^\vee/r$ is the dual of $1 \in \Omega_{N^\vee-\alpha^\vee,N^\vee}$. □

Proposition 5.39. Let r, α, and N be as in Notation 5.8, $\bar{q} \in \Omega_{N-\alpha,N}$ with $\bar{q}r \neq 1$, $\bar{q} \neq N$ and $\phi : M \longrightarrow M_{r,\alpha,N}$ be a surjective map. Set $\chi = (r, \alpha, N, \bar{q}, \phi)$. Then

(1) $\bar{q} = N/(\alpha, N)$: $\Delta^\chi = \mathcal{E}^\xi$, $\xi : M \xrightarrow{\phi} M_{r,\alpha,N} \longrightarrow M_{r,\alpha,N}/\langle m \rangle \simeq \langle n \rangle \simeq \mathbb{Z}/(\alpha, N)\mathbb{Z}$; $\bar{q}\alpha \equiv 1 \bmod N$: $\Delta^\chi = \mathcal{E}^\zeta$, $\zeta : M \xrightarrow{\phi} M_{r,\alpha,N} = \langle e_1 \rangle$;

(2) $\bar{q} = 1$: $\Lambda^\chi = \mathcal{E}^\omega$, $\omega : M \xrightarrow{\phi} M_{r,\alpha,N} \longrightarrow M_{r,\alpha,N}/\langle n \rangle = \langle m \rangle \simeq \mathbb{Z}/r\mathbb{Z}$; $w_{\bar{q}} = 1$: $\Lambda^\chi = \mathcal{E}^\theta$, $\theta : M \xrightarrow{\phi} M_{r,\alpha,N} = \langle e_2 \rangle$;

(3) $\bar{q} > 1$ and $w_{\bar{q}} \neq 1$: $\Lambda^\chi = \Delta^{r,\alpha,N,\bar{q}-\hat{q},\phi}$.

In particular, in the first two cases we have $h_{\Lambda^\chi} = h_{\Delta^\chi} = 1$. □

Proof. We can assume $M = M_{r,\alpha,N}$ and $\phi = \mathrm{id}$. The algebra associated to 0^{Λ^χ} and 0^{Δ^χ} are, respectively, $C_{\bar{q}} = k[s, t]/(s^z, t^x - s^w, s^{\hat{q}r}t^{d_{\hat{q}}} - 0^\gamma)$ and $B_{\bar{q}} = k[s, t]/(s^z - t^y, t^x, s^{\hat{q}r}t^{d_{\hat{q}}})$ by Proposition 5.35.

(1) If $\bar{q} = N/(\alpha, N)$, then $z = o(m)$, $y = 0$, $d_{\hat{q}} = (\alpha, N)$ and so $B_{\bar{q}} = k[s, t]/(s^{o(m)} - 1, t^{(\alpha,N)})$, the algebra associated to $0^{\mathcal{E}^\xi}$. If $\bar{q}\alpha \equiv 1 \bmod N$ then $r^\vee = (\alpha, N) = 1$ and $\bar{q} = \alpha^\vee/r$, that is, $\bar{q}^\vee = 1$. So $y = 1$ and $B_{\bar{q}} \simeq k[s]/(s^{|M|})$, the algebra associated to $0^{\mathcal{E}^\gamma}$.

(2) If $\bar{q} = 1$, then $z = r$, $\hat{q} = w = 0$, $x = d_{\hat{q}} = N$ and so $C_1 = k[s, t](t^n - 1, s^r)$, the algebra associated to $0^{\mathcal{E}^\omega}$. If $w = 1$ then $\bar{q} > 1$ and so $C_{\bar{q}} = k[t]/(t^{|M|})$, the algebra associated to $0^{\mathcal{E}^\theta}$.

(3) If $\bar{q} > 1$ then $H_{C_{\bar{q}}} = 0$ and so $C_{\bar{q}}$ is an algebra as in Problem 5.9. An easy computation shows that $z_{C_{\bar{q}}} = w > 1$, so that $\bar{q}_{C_{\bar{q}}} = \bar{q} - \hat{q}$ and $\lambda_{\bar{q}} = 1$. Therefore, $\Lambda^{\chi} = \Delta^{r,\alpha,N,\bar{q}-\hat{q}}$ by Theorem 5.31. ■

Proposition 5.40. $\Sigma_M^{\vee} = \Sigma_M$ and we have a bijection

$$\Delta^* : \Sigma_M/(-)^{\vee} \longrightarrow \{\text{smooth extremal rays } \mathcal{E} \text{ with } h_{\mathcal{E}} = 2\}.$$ □

Proof. $\Sigma_M^{\vee} \subseteq \Sigma_M$ since $\bar{q}\alpha \not\equiv 1 \bmod N$ is equivalent to $\bar{q}^{\vee} r^{\vee} \neq 1$. Now, let $\mathcal{E}$ be a smooth extremal ray such that $h_{\mathcal{E}} = 2$ and A the associated algebra over some field k. We can assume $H_{A/k} = H_{\mathcal{E}} = 0$. The relation $h_{\mathcal{E}} = 2$ means that there exist $0 \neq m, n \in M$, $m \neq n$ such that A is generated in degrees m, n. So $M = M_{r,\alpha,N}$ as in Notation 5.8 and A is an algebra as in Problem 5.9. By Theorem 5.31 and Proposition 5.39 we can conclude that there exist $\chi \in \Sigma_M$ such that $\mathcal{E} = \Delta^{\chi}$.

Now let $\chi = (r, \alpha, N, \bar{q}, \phi) \in \Sigma_M$. We have to prove that $h_{\Delta^{\chi}} = 2$ and, since $M_{r,\alpha,N} \neq 0$, assume by contradiction that $h_{\Delta^{\chi}} = 1$. We can assume $M = M_{r,\alpha,N}$ and $\phi = \mathrm{id}$. Note that $h_{\Delta^{\chi}} = 1$ means that the associated algebra B is generated in degree m or n. If A is an algebra as in Problem 5.9, then A is generated in degree n if and only if $z = 1$, which means $\bar{q}r = 1$. So B is generated in degree m, that is, $B^{\vee}$ is generated in degree $e_2 \in M_{r^{\vee},\alpha^{\vee},N^{\vee}}$, which is equivalent to $1 = z_{B^{\vee}} = \bar{q}^{\vee} r^{\vee} = 1$, and, as we have seen, to $\bar{q}\alpha \equiv 1 \bmod N$.

Now let $\chi' = (r', \alpha', N', \bar{q}', \phi') \in \Sigma_M$ such that $\mathcal{E} = \Delta^{\chi} = \Delta^{\chi'}$. Again we can assume $H_{\mathcal{E}} = 0$ and take B, B' the algebras associated, respectively, to χ, χ'. By definition of Δ_*, ϕ, ϕ' are isomorphisms. If $g = \phi' \circ \phi^{-1} : M_{r,\alpha,N} \longrightarrow M_{r',\alpha',N'}$ then we have a graded isomorphism $p : B \longrightarrow B'$ such that $p(B_l) = B'_{g(l)}$. Therefore, $g(\{e_1, e_2\}) = \{e_1, e_2\}$, that is, $g = \mathrm{id}$ or $g = (-)^{\vee}$. It is now easy to show that $\chi' = \chi$ or $\chi' = \chi^{\vee}$. ■

Notation 5.41. We set $\Phi_M = \{\phi : M \longrightarrow \mathbb{Z}/l\mathbb{Z} \mid l > 1, \phi \text{ surjective}\}$, $\Theta_M^2 = \{\mathcal{E}^{\phi}\}_{\phi \in \Phi_M} \cup \{(\Lambda^{\chi}, \Delta^{\chi})\}_{\chi \in \bar{\Sigma}_M}$, where $\bar{\Sigma}_M$ is the set of sequences $(r, \alpha, N, \bar{q}, \phi)$ where $r, \alpha, N \in \mathbb{N}$ satisfy $0 \leq \alpha < N, r > 0, r > 1$ or $\alpha > 1$, $\bar{q} \in \Omega_{N-\alpha,N}$ satisfy $\bar{q}r \neq 1$, $\bar{q} \neq N$ and $\phi : M \longrightarrow M_{r,\alpha,N}$ is a surjective map. Finally, set $\underline{\mathcal{E}} = (\mathcal{E}^{\phi}, \Delta^{\chi})_{\phi \in \Phi_M, \chi \in \Sigma_M/(-)^{\vee}}$. □

Theorem 5.42. Let M be a finite abelian group. Then

$$\{h \leq 2\} = \left(\bigcup_{\phi \in \Phi_M} \mathcal{Z}_M^{\mathcal{E}^{\phi}} \right) \bigcup \left(\bigcup_{(\Lambda,\Delta) \in \Theta_M^2} \mathcal{Z}_M^{\Lambda,\Delta} \right).$$

In particular $\{h \leq 2\} \subseteq \mathcal{Z}_M^{\mathrm{sm}}$. Moreover, $\pi_{\underline{\mathcal{E}}} : \mathcal{F}_{\underline{\mathcal{E}}} \longrightarrow \mathrm{D}(M)\text{-Cov}$ induces an equivalence of categories

$$\left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda) \in \mathcal{F}_{\underline{\mathcal{E}}} \,\middle|\, \begin{array}{c} V(z_{\mathcal{E}^1}) \cap \cdots \cap V(z_{\mathcal{E}^r}) \neq \emptyset \text{ iff} \\ r = 1 \text{ or } (r = 2 \text{ and } (\mathcal{E}^1, \mathcal{E}^2) \in \Theta_M^2) \end{array} \right\} = \pi_{\underline{\mathcal{E}}}^{-1}(h \leq 2) \xrightarrow{\simeq} \{h \leq 2\}. \qquad \square$$

Proof. The expression of $\{h \leq 2\}$ follows from Theorem 5.31 and Proposition 5.35. Taking into account Proposition 5.40, the last part instead follows from Theorem 3.44 taking $\Theta = \Theta_M^2$. ■

In [13], the authors prove that the toric Hilbert schemes associated to a polynomial algebra in two variables are smooth and irreducible. The same result is true more generally for multigraded Hilbert schemes, as proved later in [14]. Here, we obtain an alternative proof in the particular case of equivariant Hilbert schemes:

Corollary 5.43. If M is a finite abelian group and $m, n \in M$ then $M\text{-Hilb}^{m,n}$ is smooth and irreducible. □

Proof. Taking into account the diagram in Remark 4.10 it is enough to note that $\mathrm{D}(M)\text{-Cov}^{m,n} \subseteq \{h \leq 2\} \subseteq \mathcal{Z}_M^{\mathrm{sm}}$. ■

Proposition 5.44. $\Sigma_M = \emptyset$ if and only if $M \simeq (\mathbb{Z}/2\mathbb{Z})^l$ or $M \simeq (\mathbb{Z}/3\mathbb{Z})^l$. □

Proof. For the only if, note that if $\phi : M \longrightarrow \mathbb{Z}/l\mathbb{Z}$ with $l > 3$ is surjective, then, taking $m = l-1, n = 1 \in \mathbb{Z}/l\mathbb{Z}$, we have $\mathbb{Z}/l\mathbb{Z} \simeq M_{1,l-1,l}$ and $(1, l-1, l, 2, \phi) \in \Sigma_M$.

For the converse set $M = (\mathbb{Z}/p\mathbb{Z})^l$, where $p = 2, 3$ and, by contradiction, assume that we have $(r, \alpha, N, \bar{q}, \phi) \in \Sigma_M$. In particular, ϕ is a surjective map $M \longrightarrow M_{r,\alpha,N}$. If $e_1, e_2 \in M_{r,\alpha,N}$ are $\mathbb{F}_p$-independent then $M_{r,\alpha,N} = \langle e_1 \rangle \times \langle e_2 \rangle$, $\alpha = 0$, $\Omega_{N-\alpha,N} = \{1\}$ and therefore $\bar{q} = 1 = N/(\alpha, N)$, which implies that $\chi \notin \Sigma_M$. On the other hand, if $M_{1,\alpha,p} \simeq \mathbb{Z}/p\mathbb{Z}$, the only extremal rays for $\mathbb{Z}/p\mathbb{Z}$ are $\mathcal{E}^{\mathrm{id}}$ and, if $p = 3$, $\mathcal{E}^{-\mathrm{id}}$ since $K_{+\mathbb{Z}/p\mathbb{Z}} \simeq \mathbb{N}^{p-1}$ by Proposition 4.18. ■

Theorem 5.45. Let M be a finite abelian group and X be a locally noetherian and locally factorial scheme. Consider the full subcategories

$$\mathfrak{C}_X^2 = \left\{ (\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \delta) \in \mathcal{F}_{\underline{\mathcal{E}}}(X) \,\middle|\, \begin{array}{c} \operatorname{codim}_X V(z_{i_1}) \cap \cdots \cap V(z_{i_s}) \geq 2 \\ \text{if } \nexists \underline{\delta} \in \Theta_M^2 \text{ s.t. } \mathcal{E}^{i_1}, \ldots \mathcal{E}^{i_s} \underline{\subseteq} \underline{\delta} \end{array} \right\} \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X)$$

and

$$\mathcal{D}_X^2 = \{Y \xrightarrow{f} X \in \mathrm{D}(M)\text{-Cov}(X) \mid h_f(p) \leq 2 \ \forall \ p \in X \text{ with } \operatorname{codim}_p X \leq 1\} \subseteq \mathrm{D}(M)\text{-Cov}(X).$$

Then $\pi_{\underline{\mathcal{E}}}$ induces an equivalence of categories

$$\mathcal{C}_X^2 = \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{D}_X^2) \xrightarrow{\simeq} \mathcal{D}_X^2.$$ □

Proof. Apply Theorem 3.52 with $\Theta = \Theta_M^2$. ■

Remark 5.46. In general $\{h \leq 3\}$ does not belong to the smooth locus on $\mathcal{Z}_M$. For example, if $M = \mathbb{Z}/4\mathbb{Z}$, $\mathrm{D}(M)\text{-Cov} = \{h \leq 3\}$ is integral but not smooth by Proposition 4.18 and Remark 4.20. □

5.5 Normal crossing in codimension 1

In this subsection, we want to describe, in the spirit of classification Theorem 4.42, covers of a locally noetherian and locally factorial scheme with no isolated points and with $(\operatorname{char} X, |M|) = 1$ whose total space is normal crossing in codimension 1.

Definition 5.47. A scheme X is normal crossing in codimension 1 if for any codimension 1 point $p \in X$ there exists a local and etale map $\hat{\mathcal{O}}_{X,p} \longrightarrow R$, where R is $k[\![x]\!]$ or $k[\![s,t]\!]/(st)$ for some field k and $\hat{\mathcal{O}}_{X,p}$ denote the completion of $\mathcal{O}_{X,p}$. □

Remark 5.48. If X is locally of finite type over a perfect field k, one can show that the above condition is equivalent to having an open subset $U \subseteq X$ such that $\operatorname{codim}_X X - U \geq 2$ and there exists an etale coverings $\{U_i \longrightarrow U\}$ with etale maps $U_i \longrightarrow \operatorname{Spec} k[x_1, \dots, x_{n_i}]/(x_1 \cdots x_{r_i})$ for any i. Anyway, we will not use this property. □

Notation 5.49. In this subsection, we will consider a field k and we will set $A = k[\![s,t]\!]/(st)$. Given an element $\xi \in \operatorname{Aut}_k k[\![x]\!]$ we will write $\xi_x = \xi(x)$ so that, if $p \in k[\![x]\!]$ then $\xi(p)(x) = p(\xi_x)$. We will call $I \in \operatorname{Aut}_k k[\![s,t]\!]$ the unique map such that $I(s) = t$, $I(t) = s$. Given $B \in k^*$ we will denote by $\underline{B}$ the automorphism of $k[\![x]\!]$ such that $\underline{B}_x = Bx$.

Finally, given $f \in k[\![x_1, \dots, x_n]\!]$ and $g \in k[x_1, \dots, x_n]$ the notation $f = g + \cdots$ will mean $f \equiv g \bmod (x_1, \dots, x_r)^{\deg g + 1}$. □

The first problem to deal with is to describe the action on A of a finite group M and check when A is a $\mathrm{D}(M)$-cover over A^M, assuming to have the $|M|$-roots of unity in k. We start collecting some general facts about A.

Proposition 5.50. We have:

(1) $A = k \oplus sk[\![s]\!] \oplus tk[\![t]\!]$.

(2) Given $f, g \in A - \{0\}$ then $fg = 0$ if and only if $f \in sk[\![s]\!]$, $g \in tk[\![t]\!]$ or vice versa.

(3) Any automorphism in $\mathrm{Aut}_k A$ is of the form (ξ, η) or $I(\xi, \eta)$ where $\xi, \eta \in \mathrm{Aut}_k k[\![x]\!]$ and $(\xi, \eta)(f(s,t)) = f(\xi_s, \eta_t)$.

(4) If $\xi \in \mathrm{Aut}_k k[\![x]\!]$ has finite order then $\xi = \underline{B}$ where B is a root of unity in k. In particular, if $(\xi, \eta) \in \mathrm{Aut}_k A$ has finite order then $\xi = \underline{B}$, $\eta = \underline{C}$ where B and C are roots of unity in k.

(5) Let $f \in k[\![x]\!] - \{0\}$, B, C roots of unity in k. Then $f(Bx) = Cf(x)$ if and only if $C = B^r$ for some $r > 0$ and, if we choose the minimum r, $f \in x^r k[\![x^{o(B)}]\!]$. □

Proof. (1) is straightforward and (2) follows easily expressing f and g as in (1). For (3) note that if $\theta \in \mathrm{Aut}_k A$ then $\theta(s)\theta(t) = 0$ and apply (2). Finally (4) and (5) can be shown looking at the coefficients of ξ_x and of f. ■

Lemma 5.51. If $M < \mathrm{Aut}_k A$ is a finite subgroup containing only automorphisms of the form (ξ, η) then $A^M \simeq A$. □

Proof. It is easy to show that $A^M \simeq k[\![s^a, t^b]\!]/(s^a t^b) \simeq A$, where $a = \mathrm{lcm}\{i \mid \exists(\underline{A}, \underline{B}) \in M \text{ s.t. } \mathrm{ord} A = i\}$ and $b = \mathrm{lcm}\{i \mid \exists(\underline{A}, \underline{B}) \in M \text{ s.t. } \mathrm{ord} B = i\}$. ■

Since we are interested in covers of regular in codimension 1 schemes (and A is clearly not regular) we can focus on subgroups $M < \mathrm{Aut}_k A$ containing some $I(\xi, \eta)$.

Lemma 5.52. Let $M < \mathrm{Aut}_k A$ be a finite abelian group and assume that $(\mathrm{char}\, k, |M|) = 1$ and that there exists $I(\xi, \eta) \in M$. Then, up to equivariant automorphisms, we have $M = \langle I(\mathrm{id}, \underline{B})\rangle$ or, if M is not cyclic, $M = \langle(\underline{C}, \underline{C})\rangle \times \langle I \rangle$ where $\underline{B}$ and $\underline{C}$ are roots of unity and $o(C)$ is even. □

Proof. The existence of an element of the form $I(\xi, \eta)$ in M implies that s and t cannot be homogeneous in m_A/m_A^2, that $2 \mid |M|$ and therefore that $\mathrm{char}\, k \neq 2$.

Applying the exact functor $\mathrm{Hom}_k^M(m_A/m_A^2, -)$, we get that the surjection $m_A \longrightarrow m_A/m_A^2$ has a k-linear and M-equivariant section. This means that there exists $x, y \in m_A$ such that $m_A = (x, y)$ and M acts on x and y with characters χ and ζ. In this way, we get an action of M on $k[\![X, Y]\!]$ and an equivariant surjective map $\phi : k[\![X, Y]\!] \longrightarrow A$. Moreover, $\operatorname{Ker}\phi = (h)$, where $h = fg$ and $f, g \in k[\![X, Y]\!]$ are such that $\phi(f) = s$, $\phi(g) = t$. We can write $f = aX + bY + \cdots$, $g = cX + dY + \cdots$ with $ad - bc \neq 0$. Since $ax + by = s$ in m_A/m_A^2 and s is not homogeneous there, we have $a, b \neq 0$. Similarly, we get $c, d \neq 0$. In particular, up to normalize f, g, and x we can assume $b = c = d = 1$. Now $h = aX^2 + (a+1)XY + Y^2 + \cdots$ and applying Weierstrass preparation theorem [12, Theorem 9.2], there exists a unique $\tilde{h} \in (h)$ such that $(\tilde{h}) = (h)$ and $\tilde{h} = \psi_0(X) + \psi_1(X)Y + Y^2$. The uniqueness of $\tilde{h}$ and the M-invariance of (h) yield the relations $m(\tilde{h}) = \eta(m)^2\tilde{h}$,

$$m(\psi_0) = \psi_0(\chi(m)X) = \eta(m)^2\psi_0, \quad m(\psi_1) = \psi_1(\chi(m)X) = \eta(m)\psi_1 \tag{5.4}$$

for any $m \in M$. Moreover, $\tilde{h} = \mu h$ where $\mu \in k[\![X, Y]\!]^*$ and, since the coefficient of Y^2 in both h and $\tilde{h}$ is 1, we also have $\mu(0) = 1$. In particular, $\psi_0 = aX^2 + \cdots$ and $\psi_1 = (a+1)X + \cdots$ and so $(a+1)(\chi - \zeta) = 0$ by (5.4). Since s is not homogeneous in m_A/m_A^2, $\chi \neq \eta$ and $a = -1$. Since $\operatorname{char} k \neq 2$ we can write $\tilde{h} = (Y + \psi_1/2)^2 - (\psi_1^2/4 - \psi_0) = y'^2 - z'$. Note that y' and z' are homogeneous thanks to (5.4). Moreover, by Hensel's lemma, we can write $z' = X^2 + \cdots = X^2q^2$ for an homogeneous $q \in k[\![x]\!]$ with $q(0) = 1$. So $x' = xq$ is homogeneous and $\tilde{h} = y'^2 - x'^2$. This means that we can assume $s = x - y$, $t = x + y$. In particular, $\chi^2 = \eta^2$ and M acts on s and t as

$$m(s) = \frac{\chi + \zeta}{2}(m)s + \frac{\chi - \zeta}{2}(m)t, \quad m(t) = \frac{\chi - \zeta}{2}(m)s + \frac{\chi + \zeta}{2}(m)t.$$

Consider the exact sequence

$$0 \longrightarrow H \longrightarrow M \xrightarrow{\chi/\eta} \{-1, 1\} \longrightarrow 0. \tag{5.5}$$

If M is cyclic, say $M = \langle m \rangle$, we have $\chi(m) = -\eta(m)$ and so $m = I(\underline{B}, \underline{B})$, where $B = (\chi(m) - \eta(m))/2$ is a root of unity. Up to normalize s, we can write $m = I(\mathrm{id}, \underline{B})$.

Now assume that M is not cyclic. The group H acts on s and t with the character $\chi_{|H} = \zeta_{|H}$ and this yields an injective homomorphism $\chi_{|H} : H \longrightarrow \{\text{roots of unity of } k\}$. So $H = \langle(\underline{C}, \underline{C})\rangle$ for some root of unity C. The extension (5.5) corresponds to an element of $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, H) \simeq H/2H$ that differs to the sequence $0 \longrightarrow H \longrightarrow \mathbb{Z}/2o(C)\mathbb{Z} \longrightarrow \{-1, 1\} \longrightarrow 0$. So $H/2H \simeq \mathbb{Z}/2\mathbb{Z}$, $o(C)$ is even and the sequence (5.5) splits. We can conclude that

Table 1.

H	$m, n, r, \alpha, N, \bar{q}$	B	$\mathcal{E}$
$\mathbb{Z}/2\mathbb{Z}$	$1, 1, 1, 1, 2, 1$	$\dfrac{k[\![z]\!][U]}{(U^2 - z^2)}$	$2\mathcal{E}^{\mathrm{id}}$
$(\mathbb{Z}/2\mathbb{Z})^2$	$(1, 0), (0, 1), 2, 0, 2, 1$	$\dfrac{k[\![z]\!][U, V]}{(U^2 - z, V^2 - z)}$	$\mathcal{E}^{\mathrm{pr}_1} + \mathcal{E}^{\mathrm{pr}_2}$
$\mathbb{Z}/2l\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ $l > 1$	$(1, 0), (1, 1), 2, 2, 2l, 1$	$\dfrac{k[\![z]\!][U, V]}{(U^2 - V^2, V^{2l} - z)}$	$\Delta^{2,2,2l,1}$
$\mathbb{Z}/4l\mathbb{Z}$	$1, 2l + 1, 1, 2l + 1, 4l, 2$	$\dfrac{k[\![z]\!][U, V]}{(U^2 - V^2, V^{2l+1} - zU, UV^{2l-1} - z)}$	$\Delta^{1,2l+1,4l,2}$
$\mathbb{Z}/2l\mathbb{Z}$ $l > 1$ odd	$1, l + 1, 2, 2, l, 1$	$\dfrac{k[\![z]\!][U, V]}{(U^2 - V^2, V^l - z)}$	$\Delta^{2,2,l,1}$

$M = \langle(\underline{C}, \underline{C})\rangle \times \langle m\rangle$, where $m = I(\underline{D}, \underline{D})$ for some root of unity D and $o(m) = 2$. Normalizing s we can write $m = I(\mathrm{id}, \underline{D}) = I$. ■

Proposition 5.53. Let $M < \mathrm{Aut}_k A$ be a finite abelian group such that $(\mathrm{char}\, k, |M|) = 1$ and that there exists $I(\xi, \eta) \in M$. Also assume that k contains the $|M|$-roots of unity. Then $A^M \simeq k[\![z]\!]$, $A \in \mathrm{D}(M)\text{-}\mathrm{Cov}(A^M)$ and only the following possibilities happen: there exists a row of Table 1 such that $M \simeq H$ is generated by m, n, $H \simeq M_{r,\alpha,N}$, $A \simeq B$ as M-covers, where $\deg U = m$, $\deg V = n$ and A over A^M is given by multiplication $z^{\mathcal{E}}$. Moreover, all the rays of the form Δ^* in the table satisfy $h_{\Delta^*} = 2$. □

Proof. We can reduce the problem to the actions obtained in Lemma 5.52. We first consider the cyclic case, that is, $M = \langle I(\mathrm{id}, \underline{B})\rangle \simeq \mathbb{Z}/2l\mathbb{Z}$ where $l = o(B)$. There exists E such that $E^2 = B$. Given $0 \leq r < |M| = 2l$, we want to compute $A_r = \{a \in A \mid I(\mathrm{id}, \underline{B})a = E^r a\}$. The condition $a = c + f(s) + g(t) \in A_r$ holds if and only if $a = 0$ when $r > 0$, $f(t) = E^r g(t)$ and $g(Bs) = E^r f(s)$. Moreover, $f(t) = E^{-r} g(Bt) = E^{-2r} f(Bt) \Longrightarrow f(Bt) = B^r f(t)$. If we denote by δ_r the only integer such that $0 \leq \delta_r < l$ and $\delta_r \equiv r \bmod l$, we have that, up to constants, A^r is given by elements of the form $E^r f(s) + f(t)$ for $f \in X^{\delta_r} k[\![X^l]\!]$. Call $\beta = s^l + t^l \in A_0 = A^M$ and $v_r = E^r s^{\delta_r} + t^{\delta_r}$, $v_0 = 1$. We claim that $A^M = A_0 = k[\![\beta]\!]$ and v_r freely generates A_r as an A_0 module. The first equality holds since A_0 is a domain and we have relations

$$\sum_{n \geq 1} a_n s^{nl} + \sum_{n \geq 1} a_n t^{nl} = \sum_{n \geq 1} a_n (s^l + t^l)^n = \sum_{n \geq 1} a_n \beta^n$$

while the second claim come from the relation

$$E^r s^{\delta_r}(c+h(s)) + t^{\delta_r}(c+h(t)) = (E^r s^{\delta_r} + t^{\delta_r})(c+h(s)+h(t)) \quad \text{for } h \in X^l k\llbracket X^l \rrbracket$$

and the fact that v_r is not a zero divisor in A.

So $A \in \mathrm{D}(M)\text{-Cov}(k\llbracket\beta\rrbracket)$ and it is generated by $v_1 = Es + t$ and $v_{l+1} = -Es + t$ and so in degrees 1 and $l+1$. If $l = 1$, so that $M \simeq \mathbb{Z}/2\mathbb{Z}$, $B = 1$, $E = -1$ and $v_1^2 = \beta^2$. This means that $A \simeq k\llbracket\beta\rrbracket[U]/(U^2 - \beta^2)$ and its multiplication over $k\llbracket\beta\rrbracket$ is given by $\beta^{2\mathcal{E}^{\mathrm{id}}}$. This is the first row. Assume $l > 1$ and set $m = 1, n = l+1$. Note that $0 \neq m \neq n$ and that $M \simeq M_{r,\alpha,N}$ for some r, α, N that we are going to compute.

l odd. We have $r = \alpha = 2$ and $N = l$ since $\langle l+1 \rangle = \langle 2 \rangle \subseteq \mathbb{Z}/2l\mathbb{Z}$. Consider $\bar{q} = 1 \in \Omega_{N,N-\alpha}$ and the associated numbers are $z = r = 2, y = \alpha = 2, \hat{q} = 0, d_{\hat{q}} = x = N = l, w = 0$. Since $v_1^z = v_{l+1}^y$ and $v_{l+1}^l = \beta$, we will have $A \simeq_{k\llbracket\beta\rrbracket} A^1_{\lambda,\mu}$ where $\lambda, \mu = 1, \beta \in k\llbracket\beta\rrbracket$ (see Definition 5.29) and therefore the multiplication is $\beta^{\Delta^{2,2,l,1}}$ by Proposition 5.35. This is the fifth row.

l even. We have $r = 1, \alpha = l+1, N = 2l$ since $\langle l+1 \rangle = \mathbb{Z}/2l\mathbb{Z}$. Since $d_1 = l - 1 \equiv -\alpha$ and $d_2 = 2l - 2 \equiv 2(-\alpha)$ modulo $2l$ we can consider $\bar{q} = 2 \in \Omega_{N-\alpha,N}$. The associated numbers are $z = y = 2, \hat{q} = 1, d_{\hat{q}} = l - 1, x = N - (d_{\bar{q}} - d_{\hat{q}}) = l + 1, w = 1 \equiv xn = (l+1)^2 \bmod 2l$. Since $v_1^z = v_{l+1}^y, v_{l+1}^x = \beta v_1$ and $v_1^{\hat{q}r} v_{l+1}^{d_{\hat{q}}} = \beta$, we will have $A \simeq_{k\llbracket\beta\rrbracket} A^2_{\lambda,\mu}$ where $\lambda, \mu = 1, \beta \in k\llbracket\beta\rrbracket$ whose multiplication is $\beta^{\Delta_{1,l+1,2l,2}}$. This is the fourth row.

Now consider the case $M = \langle(\underline{C}, \underline{C})\rangle \times \langle I \rangle$ with $o(C) = l$ even. Set $\beta = s^l + t^l$, $v_{1,0} = s + t$ and $v_{1,1} = -s + t$. Note that $v_{r,i}$ is homogeneous of degree (r, i). Set $m = (1, 0), n = (1, 1)$. They are generators of M and so $M \simeq M_{r,\alpha,N}$ for some r, α, N. We have $N = o(n) = l$, $r > 1$ since $\langle n \rangle \neq M$ and so $r = 2$ since $2m = 2n$. If $l = 2$ we get $\alpha = 0$ and if $l > 2$ we get $\alpha = 2$. Choose $\bar{q} = 1$ so that the associated numbers are $z = 2, y = \alpha, \hat{q} = 0, d_{\hat{q}} = x = N = l, w = 0$. As done above, it is easy to see that $A^M = k\llbracket\beta\rrbracket$. We first consider the case $l = 2$. Since $v_{1,0}^2 = \beta$ and $v_{1,1}^2 = \beta$, we get a surjection $A^1_{\beta,\beta} \longrightarrow A$ which is an isomorphism by dimension. From the expression of $A^1_{\beta,\beta}$, we can deduce directly that the multiplication is $\beta^{\mathcal{E}^{\mathrm{pr}_1} + \mathcal{E}^{\mathrm{pr}_2}}$, where $\mathrm{pr}_i : (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ are the two projections. This is the second row.

Now assume $l > 2$. Since $v_{1,0}^2 = v_{1,1}^2$ and $v_{1,1}^l = \beta$ and arguing as above we get $A \simeq_{k\llbracket\beta\rrbracket} A^1_{\lambda,\mu}$ where $\lambda, \mu = 1, \beta \in k\llbracket\beta\rrbracket$ and the multiplication $\beta^{\Delta^{2,2,l,1}}$. This is the third row.

Finally, the last sentence is clear by definition of Σ_M and Proposition 5.40. ■

Remark 5.54. If X is a locally noetherian integral scheme and there exists a $\mathrm{D}(M)$-cover Y/X such that Y is normal crossing in codimension 1, then X is defined over a field. Indeed if $\operatorname{char} \mathcal{O}_X(X) = p$ then $\mathbb{F}_p \subseteq \mathcal{O}_X(X)$. Otherwise, $\mathbb{Z} \subseteq \mathcal{O}_X(X)$ and we have to prove that any prime number $q \in \mathbb{Z}$ is invertible. We can assume $X = \operatorname{Spec} R$, where R is a local noetherian domain. If $\dim R = 0$ then R is a field, otherwise, since $\operatorname{ht}(q) \leq 1$, we can assume $\dim R = 1$ and R complete. By definition of normal crossing in codimension 1, if $Y = \operatorname{Spec} S$ and $p \in Y$ is over m_R we have a flat and local map $R \longrightarrow S \longrightarrow S_p \longrightarrow B$, such that B contains a field k. The prime q is a nonzero divisor in R and therefore in B. In particular, $0 \neq q \in k^* \subseteq B^*$ and $q \in R^*$. □

Theorem 5.55. Let M be a finite abelian group, X be a locally noetherian and locally factorial scheme with no isolated points and $(\operatorname{char} X, |M|) = 1$. Consider the full subcategory

$$NC^1_X = \{Y/X \in \mathrm{D}(M)\text{-}\mathrm{Cov}(X) \mid Y \text{ is normal crossing in codimension } 1\} \subseteq \mathrm{D}(M)\text{-}\mathrm{Cov}(X).$$

Then $NC^1_X \neq \emptyset$ if and only if each connected component of X is defined over a field. In this case, define

$$\underline{\mathcal{E}} = \begin{pmatrix} \mathcal{E}^{\phi} \text{ for } \phi : M \longrightarrow \mathbb{Z}/l\mathbb{Z} \text{ surjective with } l \geq 1, \\ \varDelta^{2,2,l,1,\phi} \text{ for } \phi : M \longrightarrow M_{2,2,l} \text{ surjective with } l \geq 3, \\ \varDelta^{1,2l+1,4l,2,\phi} \text{ for } \phi : M \longrightarrow M_{1,2l+1,4l} \text{ surjective with } l \geq 1 \end{pmatrix}$$

and $\mathcal{C}^1_{NC,X}$ as the full subcategory of $\mathcal{F}_{\underline{\mathcal{E}}}(X)$ of objects $(\underline{\mathcal{L}}, \underline{\mathcal{M}}, \underline{z}, \lambda)$ such that:

(1) for all $\mathcal{E} \neq \delta \in \underline{\mathcal{E}}$, $\operatorname{codim} V(z_{\mathcal{E}}) \cap V(z_{\delta}) \geq 2$ except the case where $\mathcal{E} = \mathcal{E}^{\phi}$, $\delta = \mathcal{E}^{\psi}$

$$\begin{tikzcd} & & \mathbb{Z}/2\mathbb{Z} \\ M \arrow[r, two heads] \arrow[urr, bend left, "\phi"] \arrow[drr, bend right, "\psi"'] & (\mathbb{Z}/2\mathbb{Z})^2 \arrow[ur, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\ & & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

in which $v_p(z_{\mathcal{E}^{\phi}}) = v_p(z_{\mathcal{E}^{\psi}}) = 1$ if $p \in Y^{(1)} \cap V(z_{\mathcal{E}^{\phi}}) \cap V(z_{\mathcal{E}^{\psi}})$;

(2) for all $\mathcal{E} \in \underline{\mathcal{E}}$ and $p \in Y^{(1)}$ $v_p(z_{\mathcal{E}}) \leq 2$ and $v_p(z_{\mathcal{E}}) = 2$ if and only if $\mathcal{E} = \mathcal{E}^{\phi}$ where $\phi : M \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective.

Then we have an equivalence of categories

$$\mathcal{C}^1_{NC,X} = \pi^{-1}_{\underline{\mathcal{E}}}(NC^1_X) \xrightarrow{\simeq} NC^1_X.$$ □

Proof. The first claim comes from Remark 5.54. We will make use of Theorem 5.45. If $Y/X \in NC^1_Y$ and $p \in Y^{(1)}$ we have $h_{Y/X}(p) \leq \dim_{k(p)} m_p/m_p^2 \leq 2$ since etale maps preserve tangent spaces and $\dim m_A/m_A^2 \leq 2$. So $NC^1_X \subseteq \mathcal{D}^2_X$.

Let $\underline{\delta}$ be the sequence of smooth extremal rays used in Theorem 5.45. We know that $\pi_{\underline{\delta}}^{-1}(NC^1_X) \subseteq \mathcal{C}^2_X$. So we have only to prove that $\pi_{\underline{\delta}}^{-1}(NC^1_X) \subseteq \mathcal{F}_{\underline{\mathcal{E}}}(X) \subseteq \mathcal{F}_{\underline{\delta}}(X)$ and that any element $Y \in NC^1_X$ locally, in codimension 1, satisfies the requests of the theorem. Since X is a disjoint union of positive-dimensional, integral connected components, we can assume that $X = \operatorname{Spec} R$, where R is a complete discrete valuation ring. Since R contains a field, then $R \simeq k[\![x]\!]$. Let $\chi \in \pi_{\underline{\mathcal{E}}}^{-1}(\mathcal{D}^2_X)$ and D the associated M-cover over R. Let C be the maximal torsor of D/R and $H = H_{D/R}$. Note that, for any maximal ideal q of C we have $C_q \simeq k(q)[\![x]\!]$ since C/R is etale. Moreover, $\operatorname{Spec} D \in NC^1_X$ for M if and only if for any maximal prime p of D $\operatorname{Spec} D_p \in NC^1_{\operatorname{Spec} C_q}$ for M/H, where $q = C \cap p$. In the same way $\chi \in \mathcal{C}^1_{NC,X}$ for M if and only if, for any maximal prime q of C, $\chi_{|\operatorname{Spec} C_q} \in \mathcal{C}^1_{NC,\operatorname{Spec} C_q}$ for M/H. We can therefore reduce the problem to the case $H_{D/R} = 0$. We can also assume that k contains the $|M|$-roots of unity.

First assume that $\operatorname{Spec} D \in NC^1_Y$. If D is regular, the conclusion comes from Theorem 4.42. So assume D not regular and denote by $\mu : R = k[\![x]\!] \longrightarrow D$ the associated map. We know that $D/m_A = k$. By Cohen's structure theorem, we can write $D = k[\![y]\!]/I$ in such a way that $\mu_{|k} = \mathrm{id}_k$. By definition, since D is local and complete, there exists an etale extension $D \longrightarrow B = L[\![s,t]\!]/(st)$. Using the properties of complete rings, B/D is finite and so $B \simeq D \otimes_k L$. Replacing the base R by $R \otimes_k L$ we can assume that $D \simeq k[\![s,t]\!]/(st)$. The function $\mu_{|k} : k \longrightarrow D$ extends to a map $\nu : D \longrightarrow D$ sending s, t to itself. This map is clearly surjective. Since $\operatorname{Spec} D$ contains three points, ν induces a closed immersion $\operatorname{Spec} D \longrightarrow \operatorname{Spec} D$ which is a bijection. Since D is reduced ν is an isomorphism. This shows that we can write $D = A = k[\![s,t]\!]/(st)$ in such a way that $\mu_{|k} = \mathrm{id}_k$. So $\mathrm{D}(M) \simeq \underline{M}$ acts as a subgroup of $\operatorname{Aut}_k A$ such that $A^M \simeq k[\![z]\!]$. In particular, by Lemma 5.51, there exists $I(\xi, \eta) \in M$. Up to equivariant isomorphisms the possibilities allowed are described in Proposition 5.53 and coincides with the ones of the statement. So $\chi \in \mathcal{C}^1_{NC,X}$.

Now assume that $\chi \in \mathcal{C}^1_{NC,X}$. By definition of $\pi_{\underline{\mathcal{E}}}$ the multiplication that defines D over R is something of the form $\psi = \lambda z^{\mathcal{E}}$, where λ is an M-torsor and $\mathcal{E}$ is one of the ray of Table 1. The case $\mathcal{E} = \mathcal{E}^\phi$ comes from Theorem 4.42. Since, in our hypothesis, an M-torsor (in the fppf meaning) is also an etale torsor, replacing the base R by an etale neighborhood (that maintains the form $k[\![x]\!]$), we can assume $\lambda = 1$. In this case, thanks to Lemma 5.52 and Proposition 5.53, we can conclude that $A \simeq k[\![s,t]\!]/(st)$ as required. ■

Corollary 5.56. Let X be a locally noetherian and regular in codimension 1 (normal) scheme with no isolated points, M be a finite abelian group with $(\operatorname{char} X, |M|) = 1$ and $|M|$ odd. If Y/X is a $\mathrm{D}(M)$-cover and Y is normal crossing in codimension 1 then Y is regular in codimension 1 (normal). □

Proof. Since Y/X has Cohen–Macaulay fibers it is enough to prove that Y is regular in codimension 1 by Serre's criterion. So we can assume $X = \operatorname{Spec} R$, where R is a discrete valuation ring, and apply Theorem 4.42 just observing that $\widetilde{\mathrm{Reg}}^1_X = \mathcal{C}^1_{NC,X}$. ■

Remark 5.57. We keep notation from Theorem 5.55 and set $\underline{\delta} = (\mathcal{E}^\eta, \eta : M \longrightarrow \mathbb{Z}/d\mathbb{Z}$ surjective, $d > 1)$. We have that $\pi_{\underline{\delta}}^{-1}(NC^1_X) = \mathcal{C}^1_{NC,X} \cap \mathcal{F}_{\underline{\delta}}$, that is, the covers $Y/X \in NC^1_X$ writable only with the rays in $\underline{\delta}$, has the same expression of $\mathcal{C}^1_{NC,X}$ but with object in $\mathcal{F}_{\underline{\delta}}$. Therefore, the multiplications that yield a not smooth but with normal crossing in codimension 1 covers are only $\mathcal{E}^\phi + \mathcal{E}^\psi$, where ϕ and ψ are morphism as in (1), and $\mathcal{E}^{2\phi}$, where $\phi : M \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective. This result can also be found in [2, Theorem 1.9]. In particular, if $M = (\mathbb{Z}/2\mathbb{Z})^r$, where $\underline{\delta} = \underline{\mathcal{E}}$ thanks to Proposition 5.44, these are the only possibilities. □

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